ADVANCED CALCULUS - I

Time Allowed : Three Hours] [Maximum Marks : 40

Note : The candidates are required to attempt *two* questions each from Section A and B carrying 7.5 marks each and the entire Section C consisting of 10 short answer type questions carrying 1 mark each.

Section - A

1. (a) Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x+y^2}; & (x,y)\neq(0,0)\\ 0 ; & \text{otherwise}\end{cases}$$

Also discuss the continuity of f_{yx} and f_{xy} at (0, 0). 4

(b) If $f(x, y) = \begin{cases}\sin\left(\dfrac{xy}{x^2+y^2}\right); & \text{if}(x,y)\neq(0,0)\\ 0 ; & \text{otherwise}\end{cases}$

Also evaluate $f_x(0, 0)$ and $f_y(0, 0)$. If continuous at the origin. 3.5

2. (a) If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, prove that :

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$ 4

(b) Find the point on the surface $z = x^2 + y^2 + 10$ nearest to the plane $+2y - z = 0$. Use lagrange's method to find the minimum value of $x^2 + y^2 + z^2$ subject to the conditions : $x + y + z = 1$ and $xyz + 1 = 0$ 3.5

3. (a) If $Z = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, Prove that

$$x^2 Z_{xx} + 2xy\, Z_{xy} + y^2 Z_{yy} = \left(\frac{13+\tan^2 u}{144}\right)\tan u.$$ 3.5

(b) By changing the independent variables u and v to x and y using $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, show that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ transform to $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$. 4

4. (a) Obtain the second order Taylor series approximation of the function $f(x, y) = xy^2 + y\cos(x - y)$ about the point (1, 1). Find the maximum absolute error in the region $|x - 1| < 0.05$; $|y - 1| < 0.1$. 4

(b) If $V = r^m$, $r^2 = x^2 + y^2 + z^2$ then show that : $V_{xx} + V_{yy} + V_{zz} = m(m + 1)r^{m-2}$. 3.5

Section - B

5. (a) Evaluate $\iint_S \sqrt{xy - y^2}\,dx\,dy$ where S is the triangle with vertices (0, 0), (10, 1) and (1, 1). 4

(b) Evaluate $\iint r^3 dr\,d\theta$ over the area included between the circle $r = 2\cos\theta$ and $= 4\cos\theta$. 3.5

6. (a) Evaluate the intergral by changing the order of integration $\int_0^{a/\sqrt{2}} \int_0^{\sqrt{a^2-y^2}} \log(x^2 + y^2)\,dx\,dy$; $a > 0$. 3.5

(b) Evaluatge the following $\int_0^\infty \int_0^\infty \frac{e^{-y}}{y}\,dy\,dx$. 4

7. (a) Evaluate by changing to polar coordinate :
$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2 + y^2}}.$$ 4

(b) Find volume common to the cylinder $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. 3.5

8. (a) Find by the double integration the C.G. of the area of the cardioids $r = a(a + \cos\theta)$. 4

(b) Find M. I. about a-xis of the arc of the parabola $y = \sqrt{x}$ lying between (0, 0) and (4, 2). 3.5

Section - C

9. Do as directed :

(a) State Euler theorem for homogeneous function of three variables.

(b) Write formula to find area in polar form.

(c) Define saddle point of function with example.

(d) If $\phi(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$.

(e) Define maxima and minima of a function of two variable.

(f) If $x = r\cos\phi$, $y = r\sin\phi$, then evaluate $\frac{\partial(x, y)}{\partial(r, \phi)}$.

(g) Evaluate $\int_0^3 \int_0^1 (x - 2y)\,dx\,dy$.

(h) Evalute $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} = dx\,dy\,dz$

(i) Evaluate $\int_0^1 \int_{x^2}^{x} (x^2 + 3y + 2)\,dy\,dx$

(j) Define theorem of perpendicular axis on the moment of inertia of a body of mass M.

1×10=10