ADVANCED CALCULUS - I

Time : Three Hours] [Maximum Marks : 100

Note : Attempt *one* question each from Sections A, B, C, and D carrying 20 marks each and the entire Section E consisting of *eight* short answer type questions carrying 2½ marks each.

SECTION : A

1. (a) Prove that a convergent sequence is bounded. Is is converse true ? 8
   (b) Find the envelope of family of curves $x \cos^3 \theta + y \sin^3 \theta = a$ where θ is parameter. 6
   (c) Show that the function $f(x) = \frac{1}{x}$ is continuous in $[0, 1]$ but it is not uniformly continuous in $[0, 1]$. Also show that f is uniformly continuous in $[0, 1]$ where $0 < a < 1$. 6

2. (a) Prove that the sequence $\left\{\left(1+\frac{3}{n}\right)^n\right\}$ is monotonically increasing and bounded. Prove that it converges to the limit e^3. 7
   (b) Find he centre of curvature at any point (x, y) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also find evolute of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
   (c) Show that the function f defined by $f(x)=\begin{cases}1 & \text{if x is rotational} \\ 0 & \text{if x is irrational}\end{cases}$ is continuous everywhere. 6

Section : B

3. (a) Using Lagrane's Mean value theorem, show that $\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$ if $0 < u < v$. Also, deduce that $\frac{\pi}{4}+\frac{4}{41} < \tan^{-1}\frac{5}{4} < \frac{\pi}{4}+\frac{1}{8}$. 10
   (b) Discuss the continuity of the function at (0, 0) : $f(x, y)=\begin{cases}\frac{x^2}{x^4+y^2-x}, & (x, y)\neq(0, 0) \\ 0, & (x, y)=(0, 0)\end{cases}$ 10

4. (a) State and prove Taylor's theorem with Cauchy's form of remainder. 10
   (b) Let $A = \{(x, y) : x, y \in r, 0 < x < 1, 0 \leq y < 1\}$ and $f : A \to R$ defined by
   $f(x, y)=\begin{cases}1 & \text{if } y\neq 0 \\ 0 & \text{if } y=0\end{cases}$.
   Show that $\underset{(x, y)\to(\frac{1}{2}, 0)}{Lt} f(x, y)$ does not exist. 10

Section : C

5. (a) State and prove Young's theorem. 8
   (b) Prove that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent of each other unless. $\frac{a}{A}=\frac{b}{B}=\frac{h}{H}$. 6
   (c) Examine for extreme values for the function $f(x, y, z) = x^2 + y^2 + z^2 + 2xyz$. 6

6. (a) Verify Euler's Theorem for the function $z = \frac{x^{1/3} + y^{1/3}}{x^{1/4} + y^{1/4}}$. 6

(b) If $u^3 = xyz$, $\frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ and $x^2 = x^2 + y^2 + z^2$, prove that

$\frac{\partial(u, v, x)}{\partial(x, y, z)} = \frac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2x(xz+xy+yz)}$. 7

(c) `Find the extreme values (if any) of the given function $f(x, y) = x^3y^2(1 - x - y)$. 7

Section : D

8. (a) If $f(n) \geq f(n + 1) > 0 \ \forall$ n, then prove that two series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} 2^n f(2^n)$ converge or diverge together. 10

(b) Test the series $1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \ldots\ldots\ldots\infty$, $x > 0$ for convergence. 10

8. (a) Using Cauchy's Integral Test, discuss the convergence or dirergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$.

(b) State and prove D' Alembert's Ratio Test. 10

Section : E

9. Do as directed :

(a) Show that the locus of the singular points of the curves of a given family is a part of its envelope.

(b) State Sandwich Theorem.

(c) If $z = x^3 - xy + y^3$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

(d) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

(e) Wha do you mean by Alternating series ?

(f) Show that the series $\sum(-1)^{n-1}$ oscillates finitely.

(g) If $y = \sqrt{\frac{1+x}{1-x}}$, find $\frac{dy}{dx}$.

(h) Let $f : R^2 \to R$ be defined by $f(x, y) = \begin{cases} 1 & \text{if} \quad (x,y) \neq (0, 0) \\ 0 & \text{if} \quad (x,y) = (0, 0) \end{cases}$

Prove that the first order partial derivatives of f do not exist at (0, 0). 8×2½=20