ADVANCED CALCULUS–I

Semester - III

Time Allowed : Three Hours] [Maximum Marks : 35

Note : The candidates are required to attempt *two* questions each from Section A and B carrying 8 marks each and the entire Section C consisting of 8 short answer type questions carrying 1 marks.

Section : A

1. (a) If (a, b) is a point of domain of function f such that :
   (i) f_x is continuous at (a, b)
   (ii) f_y exists at (a, b). Then prove that f is differentiable at (a, b).
   (b) Show that the function :
   $$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2\neq 0\\ 0 & \text{if } x=y=0\end{cases}$$
   is continuous but not differentiable at (0, 0). 4

2. (a) Give an example to show that Young's theorem is sufficient for the equality of f_{xy} and f_{yx} at (a, b) but not necessary. 4
   (b) Prove that, by the transformations :
   $u = x - ct$, $v = u + ct$, the partial differential equation
   $\dfrac{\partial^2 z}{\partial t^2}=C^2\dfrac{\partial^2 z}{\partial x^2}$ reduces to $\dfrac{\partial^2 z}{\partial u\,\partial v}=0$, where z is a function of u and v. 4

3. State Taylor's theorem for functions of two variables. Expand $x^4 + x^2y^2 - y^4$ about the point (1, 1) upto terms of second degree. Find the form of R_2, the remainder after 2 terms. 8

4. Show that the minimum and maximum values of function :
   $f(x, y, t) = (ax + by + cz)\,e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$ are
   $-\sqrt{\frac{1}{2e}\left(a^2\alpha^{-2}+b^2\beta^2+c^2\gamma^{-2}\right)}$ and $\sqrt{\frac{1}{2e}\left(a^2\alpha^{-2}+b^2\beta^2+c^2\gamma^{-2}\right)}$ 8

Section : B

5. (a) Find the value of $\oint_c x^2y\,dx + xy^2\,dx$ taken in clockwise sense along the hexagon C whose vertices are :
   $(\pm 3a, 0)$, $(\pm 2a, \pm\sqrt{3}a)$. 6
   (b) For a function $f(x, y)=\begin{cases}\frac{1}{2}, & y \text{ rational}\\ x, & y \text{ irrational}\end{cases}$, $\int_0^1 dx\int_0^1 f\,dy$ does not exist. 2

6. State Green's theorem in plane and hence find the line integral $\int_C \dfrac{ydy - ydx}{x^2+y^2}$ taken in the positive direction over any closed contour C with origin inside it. 8

7. State Stoke's theorem and get Green's theorem as its special case. Use Stoke's theorem to show that :
   $\iint_C (y-z)\,dy\,dz+(z-x)\,dz\,dx+(x-y)\,dx\,dy=\pi a^3$ where S is the portion of the surface $x_2 + y_2 - 2ax + az = 0$, $x \geq 0$. 8

8. (a) Compute $\int_0^a dx \int_0^x dy \int_0^y xyz\,dz$. 2

(b) Compute the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. 6

Section : C

9. Do as directed : (a) If $x=\gamma \sin\theta \cos\phi$, $y=\gamma \sin\theta \sin\phi$ and $z=\gamma\cos\theta$, find $\frac{\partial(x,y,z)}{\partial(\gamma,\theta,\phi)}$

(b) Investigate for continuity of the function : $f(x,y)=\begin{cases} x^2+2y^2, & (x,y)\neq(1,2) \\ 0, & (x,y)=(1,2) \end{cases}$ at (1, 2).

(c) Write the physical interpretation of curl.

(d) Write the statement fo Gauss's Divergence theorem.

(e) Define del of a vector valued function.

(f) Show that for differentiable function f, if $z = f(x^2y)$, then $x\frac{\partial z}{\partial x}=2y\frac{\partial z}{\partial y}$.

(g) Define homogeneous function of two variables of degree n. Give one example.

(h) State necessary and sufficient conditions for a function f of two variables to have extreme value at (a, b).

1×8=8