NUMBER THEORY - II (iv)

Semester-VI

Time Allowed : Three Hours **Maximum Marks : 40**

Note : Attempt five questions in all. Select two questions each from Section A and B while Question No. IX (Section-C) is compulsory.

SECTION-A

I. Prove that the equation $x^4 + y^4 = z^2$ has no solution in positive integers. 7.5

II. (a) For any irrational number ξ show that there exist infinitely many rational number $\frac{h}{k}$ such that $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}\,k^2}$ 4, 5

(b) Write Farey fraction for F_8 and hence solve the Diophantine equation $8x + 5y = 2$. 3

III. (a) Show that the Diophantine equation

$x^4 + x^3 + x^2 + x + 1 = y^2$

has the integral solution $(-1, 1)$, $(0, 1)$, $(3, 11)$ and no other. 3, 5

(b) If d is a positive integer not a perfect square, then show that $h_n^2 - dk_n^2 = (-1)^{n-1} q_{n+1}$ for all integers $n \geq -1$. 4

IV. Prove that π is irrational number. 7.5

SECTION–B

V. (a) State and prove Euler summation formula. 5

(b) Find the least positive integer represented by $7x^2 + 25xy + 23y^2$. 2.5

VI. (a) Show that no integer of the form $4^n(8m + 7)$ can be represented as the sum of three squares. 3.5

(b) Find all integers n such that

(i) $\phi(n) = n/2$, (ii) $\phi(n) = 12$. 4

VII. Prove that for divisor function d(n), for all $n \geq 1$

$$\sum_{n \leq x} d(n) = x \log x + (2C - 1)x + O\left(\sqrt{x}\right),$$

where C is the Euler's constant.

Also find the average order of d(n). 7.5

VIII. If p is an odd prime such that $p \equiv 3 \pmod 4$ then show that p is a sum of four squares. 7.5

SECTION–C

IX. Attempt all the following :

(a) Define Pell's equation.

(b) Show that two distinct infinite simple continued fractions converge to different values.

(c) Find the two representations of $\frac{77}{30}$ as a simple continued fraction.

(d) Find all the solutions of $7x + 13y = 5$, if they exist.

(e) Show that 2^n is sum of four squares, for any $n \in Z^+$.

(f) Verify $p_m(n) = p_{m-1}(n) + p_m(n - sm)$ for $n = 8$, $m = 5$.

(g) State Abel's Identity.

(h) Prove that if f is multiplicative then $f(1) = 1$.

(i) Prove that $O(f) = O(g)$ iff $f = O(g)$.

(j) Define the average order of arithmetical function. (1×10=10)