ADVANCED CALCULUS - I

Time : Three Hours **Maximum Marks : 100**

Note :- Attempt one question each from Section A, B, C and D carrying 20 marks each and the entire Section A consisting of 10 short answer type questions carrying 2 marks each.

Section-A

1. (a) Let $A = \{(x, y) : 0 < x < 1, x, y \in R\}$

Let $f : A \to R$ be defined as $f(x, y) = x + y$.

Show that by definition $\lim_{\substack{x \to 0 \\ y \to \frac{1}{2}}} f(x, y) = \frac{1}{2}$ 6

(b) If $z = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, prove that

$x^2 z_{xx} + y^2 z_{yy} + 2xyz_{xy} = (1 - 4\sin^2 z) \sin 2z$. 6

(c) Prove that

$$e^{ax} \sin by = by + abxy + \frac{e^{atx}}{b}\left[\left(a^3x^3 - 3ab^2xy^2\right)\right] \sin bty$$

$+\left(3a^2bx^2y - b^3y^3\right)$ cos bty]

where $0 < t < 1$ 8

2. (a) Show that of all the trinagles inscribed in a circle that one with maximum area is equilateral. 6

(b) If $u^3 = xyz, \frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ and $w^2 = x^2 + y^2 + z^2$,

prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\frac{v(y-z)(z-x)(x-y)(x+y+z)}{3u^2w(yz+zx+xy)}$$ 6

(c) State and prove Taylor's theorem for 2 variables. 8

Section-B

3. (a) Evaluate

$$\int_R (x+y)^2\, dx\, dy$$

Where R is the parallelogram in the xy-plane with vertices (1,0), (3 3,1) (2, 2), (0, 1) using transformation $u = x + y,\ v = x - 2y$. 8

(b) Find Centroid of the region R bounded by $z = 4 - x^2$ and planes $x = 0, y = 0, y = 6, z = 0$. 6

(c) Evaluate $\iint xy\,(x+y)\, dx\, dy$ over $y = x^2$ and $y = x$. 6

4. (a) Change the order of integration $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ hence evaluate. 6

(b) Find volume of cylinder $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$. 6

(c) Evaluate $\iint_S \sqrt{xy - y^2}\; dx\, dy$, where S is a triangle with vertices (0, 0), (10, 1) and (1, 1) 8

Section-C

5. (a) If $\vec{F}$ is a solenoidal vector, show that

curl. curl. curl. curl $\vec{F} = \tilde{N}^4 \vec{F}$ 6

(b) Verify divergence theorem for $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. 10

(c) What is the greatest rate of increase of $f = x^2 + yz^2$ and (1, -1, 3) ? 4

6. (a) Show that $\nabla^2\left(\frac{x}{r^3}\right) = 0$, where $r = \sqrt{x^2 + y^2 + z^2}$ 4

(b) Verify Stoke's theorem for $\vec{F} = y\hat{i} - 2\hat{j} + x\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$, and C is its boundary. 10

(c) Evaluate $\int x\, dy - y\, dx$, around the circle $x^2 + y^2 = 1$ 6

Section-D

7. (a) Prove that set of all real numbers is uncountable. 10

(b) State and prove Heine Borel theorem. 10

8. (a) Show that $f(x) = x^2$ is uniformly continuous on (0, 1]. 10

(b) Prove that every infinite set contains a countable set. 10

Section-E

9\. Do as directed.

(a) If $z = e^{ax+by} f(ax - by)$, find $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$

(b) State and prove theoram on Homogeneous functions.

(c) Find total derivative of $\tan^{-1}\frac{y}{x}$.

(d) Evaluate $\int_0^1\int_0^3 (x+1)dy\,dx$

(e) Write formula for the C.G. $(\bar{x}, \bar{y}, \bar{z})$ of a solid.

(f) Find angle between the tangents to the curve $\bar{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$ at points $t = \pm 1$.

(g) Show that $\bar{A} = 3x^2y\hat{i} + (x^3 - 2yz^2)\hat{j} + (3z^2 - 2y^2z)\hat{k}$ is irrotational.

(h) Prove that curl. grad $\bar{\psi} = 0$

(i) Define Adherent point.

(j) Define partition of a set. (2×10=20)