ANALYSIS - II

Time : Three Hours **Maximum Marks : 100**

Note :- The candidates are required to attempt one question each from Sections A, B, C and D carrying 20 marks each and the entire Section E Consisting of 8 short answer type questions carrying 2½ marks each.

Section-A

1. (a) State and prove Darboux's Theorem.

(b) Let f be a function defined on $\left[0, p/4\right]$ by

$$f(x)=\begin{cases}\sin x, & \text{if } x \text{ is irrotational}\\ \cos x, & \text{if } x \text{ is rotational}\end{cases}$$

Show that f is not Riemann Integrable.

(c) Let f be Riemann Integrable on [a, b] and g be another function obtained by altering the value of f at a finite number of points of [a,b], then prove that g is also Riemann Integrable on [a, b] and $\int_a^b f\,dx = \int_a^b g\,dx$ 7+7+6

2. (a) If a function f defined on an interval [a, b] is bounded and has a finite points of discontinuity then prove that f is Riemann Integrable.

(b) The necessary and sufficient condition for a bounded function f to be Riemann Integrable on an interval [a, b] is that to every $\hat{\mathrm{I}} > 0$, however small, there exists a partition P such that U(P, f) - L(P, f) < I

(c) Evaluate $\int_0^{\pi/2} \sin x\,dx$ by calculating upper and lower integrals. 7+7+6

Section-B

3. (a) Prove that $\lim_{n\to\infty} \sqrt[n]{a}=1$ for all a > 0

(b) Prove that the sequence $\left\{\left(1+\frac{3}{n}\right)^n\right\}$ is monotonically increasing and bounded, also show that it converges to the limit e^3.

(c) Prove that the sequence $[x_n]$ converges to a real number l iff $[x_n]$ is bounded and l is the only cluster point of $[x_n]$. 7+7+6

4. (a) State and prove Leibnitz test.

(b) Using Cauchy's Integral test, discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 0$$

(c) Discuss the convergence of the series.

$$\sum \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....+\frac{1}{n}\right)\sin\,(n\theta+a)$$ 7+7+6

Section-C

5. (a) Test the sequence $\left[\frac{n^2x}{1+n^3x^2}\right]$ for uniform convergence on the interval (0, 1)

(b) Examine for term by term integration the ries

(c) If $na_{n\circledR 0}$ and f(x) $\sum a_n x^n \to s$ as $x \to 1-$, then $\sum a_n$ converges to the sum S. 7+7+6

6. (a) State and prove Abel's test.

(b) Show that tan^{-1}x = $x = \frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+......$ for - 1 £ x £ 1.

Also deduce that series $\frac{p}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}.....$

(c) Test the series $\sum nx(1-x)^4$, $0 \le x \le 1$ for uniform convergence and continuity of the sum function for all values of x. 7+7+6

Section-D

7. (a) Find all the circumcentre of the triangle whose vertices are complex numbers z_1, z_2, z_3.

(b) Find all bilinear transformation which transform the plane $I_m(z)^3$ 0 into the unit circle $|w| \le 1$

(c) If f(z) = u + iv is an analytic function of z = x + iy and u - v = e^x (cos y - sin y), find f(z) in terms of z. 7+7+6

8. (a) Show that locus of complex nmber z such that $|z - a|\,|z+a| = a^2$, $a > 0$ is a lemiscate.

(b) Let f(z) be an analytic function of z in a region D of the z-plane and let f '(z) $\neq$ 0 inside D. Then the mapping w = f(z) is conformal of the points of D.

(c) Show that w = sinh z satisfies Cauchy-Riemann equations and find $\frac{dw}{dz}$

Section-E

9. Do as directed :

(a) Evaluate $\int_0^3 f(x)\,dx$ where $f(x) = \begin{cases} x & \text{on } [0,1] \\ [x] & \text{on } [1,3] \end{cases}$

(b) Prove that $1 \le \int_0^{\pi/2} \sqrt{\sin x}\,dx \le \frac{1}{2}\sqrt{2\pi}$

(c) Give an example to show that $\{x_n \ y_n\}$ can be convergent without $\{x_n\}$, $\{y_n\}$ being convergent.

(d) Examine the convergence of the series $\sum \frac{n^2}{n!}$

(e) Find the radius of covnergence and interval of convergence of the series $\sum \frac{(x-1)^n}{2^n}$

(f) What is the difference between Pointwise convergence and Uniform convergence ?

(g) Show that $f(z) = z^2$ is uniformly continuous in the region.

(h) Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 4\frac{\partial^2 f}{\partial z \partial \overline{z}}$