MECHANICS - AND VECTOR CALCULUS–III

Time : Three Hours **Maximum Marks : 100**

Note : Attempt one question each from Sections A, B, C and D carrying 20 marks each, and the entire Section E is compulsory consisting of 8 short answer questions carrying 2½ marks each.

SECTION–A

I. (a) A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely joined together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is $W\sqrt{3}$.

(b) A heavy uniform string of length l, suspended from a fixed point A and its other end B is pulled by a horizontal force equal to the weight of length a of the string. Show that the horizontal and vertical distances between A and B are $\left[a\sinh^{-1}\left(\frac{l}{a}\right)\right]$ and $\left[\sqrt{l^2+a^2}-a\right]$ respectively. 10, 10

II. (a) A smooth cone of weight W stands inverted in a circular hole with its axis vertical. A string is wrapped twice round the cone just above the hole and pulled tight. What must be the tension in the string so that it will raise the cone ?

(b) If a system of forces acting on a body be in equilibrium and the body undergoes a slight displacement consistent with the geometrical conditions of the system. Prove that the algebraic sum of the virtual works done by the forces is zero. 10, 10

SECTION – B

III. (a) Show that a given system of forces can be replaced by two forces, equivalent to the given system, in an infinite number of ways and that the tetrahedron formed by the two forces is of constant volume.

(b) A solid hemisphere rests on a plane inclined to the horizon at an angle $\sin^{-1}\left(\frac{3}{8}\right)$. The plane is rough enough to prevent sliding. Find the position of equilibrium and show that it is stable. 10,10

IV. (a) Find the expressions for velocity along and perpendicular to the radius vector of a particle moving along a plane curve.

(b) A particle m is attached to a light wire which is stretched tightly between two fixed points under tension T. If a, b be the distances of the particle from the two ends, prove that the period of transverse oscillation is $2\pi\sqrt{\frac{mab}{T(a+b)}}$. 10, 10

SECTION–C

V. (a) A bead is constrained to move on a smooth wire in the form of an equiangular spiral. It is attached to pole of the spiral by a force $m\mu$ (distance)$^{-2}$ and starts from rest at a distance b from the pole. Show that if the equation of the spiral be $r = a e^{\theta\cot\alpha}$, the time of arriving at the pole is $\frac{\pi}{2}\sqrt{\frac{b^3}{2\mu}}\sec\alpha$.

(b) Find the law of force to the focus when the orbit is an ellipse. Find the velocity and periodic time. 10, 10

VI. (a) A particle slides down a rough curve under gravity in a vertical plane. Discuss its motion.

(b) A smooth helix is placed with its axis vertical and a small bead slides down it under gravity, show that it makes its first revolution from rest in time $2\sqrt{\frac{\pi a}{g \sin\alpha \cos\alpha}}$, where α is the angle of the helix. 10, 10

SECTION – D

VII. (a) The necessary and sufficient condition for a vector function $\vec{f}$ of a scalar variable t to have constant direction is that $\vec{f} \times \frac{d\vec{f}}{dt} = 0$.

(b) Find the value of $\nabla\left(\nabla.\left(\frac{\vec{r}}{r}\right)\right) = \frac{-2}{r^3}\vec{r}$ 10,10

VIII. (a) State and prove Guass's Divergence theorem.

(b) Evaluate by Stroke's theorem $\oint_c (e^x dx + 2y\,dy - dz)$, where C is the curve $x^2 + y^2 = 4$, $z = 2$. 10, 10

SECTION – E

IX. Do as directed :

(a) State the principle of Virtual Work.

(b) Define (i) Wrench (ii) Poinsot's central axis.

(c) Prove that the Simple Harmonic Motion is periodic with period $\frac{2\pi}{\sqrt{\mu}}$

(d) State (i) Hooke's law, (ii) Terminal velocity.

(e) If $\bar{R}$ is a unit vector in the direction of $\vec{r}$, then show that $\bar{R} \times d\bar{R} = \frac{\vec{r} \times d\vec{r}}{r}$.

(f) Define (i) Solenoidal vector (ii) Irrotational vector.

(g) Show that the vector

$(\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$ is irrotational.

(h) A particle moves vertically downwards under gravity in a resisting medium, resistance being kv^n per unit of mass. Find the maximum velocity of the medium. $8\times2\frac{1}{2} = 20$