ADVANCED CALCULUS

Paper- I
Semester-III

Time Allowed : 3 Hours] [Maximum Marks : 36

Note : The candidates are required to attempt two questions each from Section A and B carrying 5.5 marks each and the entire Section C consisting of 7 short answer type questions carrying 2 marks each.

1. (a) If $u = x^2 \tan^{-1}\left(\frac{x}{y}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

   (b) $u = f(r)$ where $r^2 = x^2 + y^2$, show that :
   $$\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} = f(x)'' = \frac{1}{r}f'(r).$$

2. (a) Show that the function $f(x, y) = |x| + |y|$ is continuous at (0, 0), but not differentiable at (0, 0).

   (b) State and prove Taylor's theorem for functions of two variables.

3. (a) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, prove that
   $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(1 - 4\sin^2 u\right)\sin 2u.$$

   (b) Show that $f(x, y) = \sin x + \cos y$, is differentiable at every point of R^2.

4. (a) Find the minimum value of $x^2 + y^2 + z^2$, when $yz + zx + xy = 3a^2$, where a is constant.

   (b) Find $\frac{\partial(y_1 y_2 \ldots\ldots y_n)}{\partial(x_1 x_2 \ldots\ldots x_n)}$, it is being given that :
   $y_1 = x_1(1 - x_2)$, $y_2 = x_1x_2(1 - x_3)$, $y_3 = (x_1x_2x_3)(1 - x_4)$ and
   $y_{n-1} = x_1x_2x_3x_4 \ldots\ldots x_{n-1}(1-x_n)$, $y_n = x_1x_2x_3x_4 \ldots\ldots x_n$.

Section - B

5. (a) Evaluate by changing the order of integration of : $\int_0^1 \int_x^{y^2-x^2} \frac{x\,dy\,dx}{\sqrt{x^2 + y^2}}$.

   (b) Evaluate $\iiint (x^2 + y^2)\,dx\,dy\,dx$ over the region bounded by $x^2 + y^2 + z^1 = 1$.

6. (a) Evaluate $\iint (a^2 - x^2 - y^2)\,dx\,dy$ over the semi circle $x^2 + y^2 \le ax$ in the first quadrant, $a > 0$.

   (b) Using triple integration, find the volume of a sphere $x^2 + y^2 + z^2 = a^2$, where a is constant.

7. (a) Using double integration, find the area of the region enclosed by the curves $y^2 = x^3$ and $y = x$.

   (b) Find the centriod of an ellipse with mass density 1.

8. (a) Find the moment of inertia of the ellipsoid
   $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$. With uniform mass density 1 about the y-axis.

   (b) Find the area of the circle of radius r by using polar co-ordinates.

Section - C

9. (a) Using definition prove, that $\lim_{(x,y)\to(1,2)} (3xy + 5x - 4) = 7$.

   (b) Find the percentage error in calculating the area of a rectangle when an error of 2 percent is made in measuring its sides.

(c) Define moment of inertial and centre of gravity in R^3.

(d) Show that : $2 \le \iint_{\substack{1 \le x \le 2 \\ 1 \le y \le 2}} \left(x^2 + y^2\right) dx\, dy \le 8$.

(e) Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right) dx\, dy$, wherre E is the region bounded by the coordinate axes and $x + y = 1$ int he first quadrant.

(f) Prove that $u = x - 2y + z$, $v = x^2 + 2xy - xz$, = 2 = $3x + 2y - z$ are not functionally dependent.

(g) State Euler's theorem of Homogeneous functions. Also define Homogeneous functions.

$7 \times 2 = 14$