MATHEMATICAL METHODS - II

Time : Three Hours] **[Maximum Marks : 100**

Note :- Attempt *one* question each from Sections A, B, C and D carrying 20 marks each. Section E is compulsory consisting of 10 short answer type questions carrying 2 marks each.

Section-A

1. (a) Let f be bounded and integrable in $[-\pi, \pi]$ and monotonic in $[-\alpha, 0)$ and $(0, \alpha]$, (not necessarily in the same sense), where δ is some positive number less than π. Prove that

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{f(+0)+f(-0)}{\pi} \int_0^{\infty} \frac{\sin x}{x}\, dx$$

where a_n, n = 0, 1, 2, ... denote the Fourier's coefficients of f.

(b) Obtain a Fourier cosine series of the periodic x^2, $0 \le x < \pi$, of period π. 10, 10

II. (a) Find the Fourier series generated by the periodic function $|x|$ of period 2π. Also compute the value at 0.

(b) if f is a bounded and integrable on $[-\pi, \pi]$ and a_n, b_n are its Fourier coefficients, then show that $\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$ converges. 10, 10

Section-B

III. (a) Evaluate : (f (t)) when

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ (t-1)^2, & t > 1 \end{cases}$$

(b) Let f(t) be real and piecewise continuous function on each interval in $[0, \infty)$ and is of exponential order α. If $L(f(t)) = F(s)$, then prove that

$$L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(s)\, ds$$

if integrla is convergent. 10, 10

IV. (a) Prove that

$$\int_0^{\pi/2} \sqrt{\tan t}\ dt = \sqrt{\frac{\pi}{2}}$$

(b) Apply convolution theorem to find inverse Laplace transform of $\frac{s^2}{(s^2+a^2)^2}$ 10, 10

Section-C

V. (a) State and prove convolution theorem for the Fourier Transforms.

(b) Evaluate $F_c^{-1}\left(\frac{1}{s^2+1}\right)$

VI. (a) Find finite Fourier cosine transform of f (t) = sin α t, t (0, π).

(b) Let

$$f(x)=\begin{cases}2, & 0 \le x \le \pi \\ 0, & x > \pi\end{cases}$$

Find Fourier sine integral of f(x) and evaluate.

$$\int_0^{\infty} \frac{1-\cos s\pi}{s} \sin s\, x\, ds$$ 10, 10

Section-D

VII. (a) Solve $(D^2 + 1)\, x = \sin 2t \sin t$, $t > 0$ when $x = 1$, $Dx = 0$ at $t = 0$.

(b) Find the solution of one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2};x,t > 0$$

where $y(x, 0) = 0$, $y_t(x, 0) = 0$, $y(0, t) = f(t)$ and $y(x, t) \to 0$ as $x \to \infty$. 10, 10

VIII. (a) Solve $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$ with the following conditions

(i) $V_x(0, t) = 0$ for $t > 0$,

(ii) $V(x,0)=\begin{cases}x, & 0 \le x \le 1 \\ 0, & x > 1\end{cases}$

(iii) V (x, t) is bounded.

(b) Solve $\frac{dx}{dt} + x - y = 0$, $\frac{dy}{dt} + x + y = 0$, given $x(0) = 1$, $y(0) = 0$. 10, 10

Section-E
(Compulsory Question)

IX. (a) Define Fourier transform and inverse Fourier transform of a function.

(b) Find Fourier sine transform of

$$f(t)=\begin{cases}2, & 0 \le t \le a \\ 0, & t > a\end{cases}$$

(c) Prove that for $t \ge 0$, $L(1) = \frac{1}{s}$, $S > 1$.

(d) Using First Shifting Theorem, prove that for

$t \ge 0$, $(L(e^{\alpha t}) = \frac{1}{s-\alpha}$, $s > \alpha$.

(e) Find Fourier transform of

$$f(t)=\begin{cases}\frac{\sqrt{2\pi}}{2a}, & -a < t < a \\ 0, & t < -a \text{ or } t > a\end{cases}$$

(f) If F(s) is Laplace tranasform of f(t) for $t \geq 0$ and α is any number prove that $F(s-\alpha)$ is Laplace transform of $e^{\alpha t}$ f(t).

(g) If F(s) is complex Fourier transform of f(t) then prove that

$$F(\cos(l\,t)\,f(t)) = \frac{F(s+l) + F(s-l)}{2}$$

(h) Find $L^{-1}\left(\frac{1}{s^{9/2}}\right)$

(i) Prove that $\beta(m, n)$ is symmetric. Also write the value of $\Gamma\left(\frac{1}{2}\right)$

(j) Express $\int_0^1 x^{-2/3}(1-x)^{1/2}\,dx$ as Beta function. (10×2=20)