NUMBER THEORY - III
Opt. (i)

Time : Three Hours] **[Maximum Marks : 100**

Note :- Attempt *one* question each from Section A, B, C and D carrying 20 marks each. Section E consisting of 8 short answer type questions carrying 20 marks in all is compulsory.

Section-A

I. (a) State and prove fundamental Theorem of Arithmetic.

(b) A man has 987 orenges. In how many ways can he distribute these among 13 women and 5 children, if all women receive equal number of oranges and all children equal number of them ?

(c) If a is any integer and n is a positive integer such that (a, n) = 1, then prove that $a^{\phi(n)} \equiv 1 \pmod{n}$. (10, 5, 5)

II. (a) If a is selected at random from {1, 2,........., 14} and b is selected from {1, 2, ..., 15} what is the probability that $ax \equiv b \pmod{15}$ has at least one solution ?

(b) If d(n) denotes the number of positive divisors of n > 1, then prove that product of positive divisors of n is $n^{\frac{d(n)}{2}}$.

(c) What is the remainder when 3^{287} is divided by 23 ? (10,5, 5)

Section-B

III. (a) Let F and f are two Arithmetic functions, for every integer n, if $F(n)=\sum_{d/n} f(d)$ then prove that $f(n)=\sum_{d/n}\mu(d)F\left(\frac{n}{d}\right)$

(b) If a has order h modulo m, then a^k has order $\frac{h}{(h, k)}$ modulo m, where k is a positive integer. (12, 8)

IV. (a) Let (a, m)= 1, then a is a primitive root of m if and only if $a^{\frac{\phi(m)}{p}} \neq 1 \pmod{m}$ for every prime divisor p of $\phi(m)$.

(b) State and prove Gauss's Lemma. (10, 10)

Section-C

V. (a) Prove that the value of a continued fraction is less than every even convergent and greater than every odd convergent.

(b) Find the successor term of $\frac{28}{29}$ in F_{3451}.

(c) Find all solutions of $x^2 + y^2 = z^2$ such that $0 < z < 30$. (10,5,5)

VI (a) Show that every rational number can be expressed as a finite simple continued fraction.

(b) Show that two consecutive terms of $F_n(n^1 \quad 1)$ cannot have two same denominators.

(c) Expand $\sqrt{3}$ as infinite continued fraction. (10,5,5)

Section-D

VII. (a) Prove that all forms of discriminant -7 are equilant Hence show that an odd prime $p \neq 3$ is representable by $x^2 + xy + 2y^2$ if and only if $p = 7$ or $p = 1,\ 2$ or $4 \pmod 7$

(b) Prove that average order of $s(n)$ is $\dfrac{p^2 n}{12}$ (12,8)

VIII. (a) Prove that the following relations are equivalent.

(i) $$\text{Lim}\left(\frac{\pi(x)\log x}{x}\right)=1 \text{ as } x \to \infty$$

(ii) $$\text{Lim}\left(\frac{\theta(x)}{x}\right)=1 \text{ as } x \to \infty$$

(iii) $$\text{Lim}\left(\frac{\psi(x)}{x}\right)=1 \text{ as } x \to \infty$$

(b) Prove that no integer of the gorm $8k + 7$ can be expressed as sum of three square.

Section-E

IX. Do as directed

(a) Shwo that $\left[\frac{[a]}{b}\right]=\left[\frac{a}{b}\right]$, where stands for integral pert function. (2)

(b) State Chinese Remainder Theorem (2)

(c) Find all integer n for which $n^2 + n + 41$ is a perfect square (2)

(d) Let $s(n)=$ sum of diviisors of n. Show that $s(n)$ is multiplicative. (2)

(e) Evaluate $\left(\frac{-42}{61}\right)$ (3)

(f) State Euler's Summation Formula

(g) Find all quadratic residues of 11 (3)

(h) Define Partition and list all partitions for $n = 4$ (3)