ANALYSIS-II

Paper-V (Semester-IV)

Time Allowed : Three Hours **Maximum Marks : 36**

Note : Attempt two questions each from Sections A and B carrying 5½ marks each, and the entire Section C consisting of 7 short answer type questions carrying 2 marks each.

SECTION-A

I. (a) Prove that a monotonically decreasing sequence $\{a_n\}$ converges if and only if it is bounded below. The limit of $\{a_n\}$, when it converges, is the greatest lower bound of $\{a_n\}$. 3

(b) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{a^n}{a^n + x^n}$ where $x > 0$, $a > 0$.

II. (a) Apply Cauchy's General Principle of convergence to show the convergence of the sequence $\{a_n\}$ where

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\frac{1}{(n-1)^2} + \frac{1}{n^2}.$$ 3

(b) Discuss the convergence of the series $\sum \frac{1,3,5\ldots\ldots(2n-1)}{2,4,6\ldots\ldots(2n)} \cdot \frac{1}{n^{\alpha}}$ where α is constant. 2½

III. (a) Prove that every sequence contains a monotone subsequence. 3

(b) Show that for the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\ldots$$

Cauchy's Root Test indicates convergence, while D'Almert's Ratio Test is Inconclusive. 2½

IV. (a) Let $\{a_n\}$ be a sequence of positive real numbers such that $a_{n+1} = (a_n + a_{n-1})$ for all $n \geq 2$. Then prove that $\{a_n\}$ converges to $(a_1 + 2u_2)$. 3

(b) Prove that the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$ is convergent for $-1 \leq x \leq 1$. 2½

SECTION-B

V. (a) Test the sequence $\left\{\frac{nx}{1+n^2x^2}\right\}$ for uniform convergence on the interval [0 1]. 3

(b) If $na_{n\to 0}$ and $f(x) = \Sigma a_n x^n \to s$ as $x \to 1-$, then Σa_n converges to the sum s. 2½

VI. (a) Test the series $\sum \frac{nx}{1+n^2x^2} + \frac{(n-1)x}{1+(n-1)^2x^2}$ for uniform convergence and continuity of the sum function for all values of x. 3

(b) Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$ for $-1 \le x \le 1$. Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ 2½

VII. (a) Test the sequence $\{xe^{-nx}\}$ for uniform convergence on the interal [0 1]. 3

(b) Let $\sum_{0}^{\infty} a_n x^n$ be a power series with unit radius of convergence nad let $f(x) = \sum_{0}^{\infty} a_n x^n$, $-1 < x < 1$. If the series Σa_n converges then prove that $\lim_{x\to 1} f(x) = \Sigma a_n$. 2½

VIII. (a) Show that $\int_0^x \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{(n+1)}}{n(n+1)}$ for $|x| < 1$. 3

(b) Examine for term by term integration the series $\Sigma x^{n-1}(1 - 2x^n)$ for $0 \le x \le 1$. 2½

SECTION–C

IX. (a) Discuss the convergence of the series $\Sigma\, r^{\log n}$ where $r > 0$.

(b) Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} (n-2)!\, x^{n-2}$.

(c) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) Show that $\left\{\frac{n}{a^n}\right\}$ is a null sequence where $\alpha > 1$.

(e) If $a_n = 1 + \frac{(-1)^n}{2n}$ find a positive integer m such that $|a_n - 1| < \frac{1}{10^3}$ for all $n \ge m$.

(f) Discuss the convergence or divergence of the series $\sum \sqrt{n^2+1} - n$.

(g) Show that the series $\sum \frac{x^n}{n!}$ converges absolutely for all x. (7×2=14)