ALGEBRA- II

Paper - III
(Semester - VI)

Time Allowed : 3 Hours] [Maximum Marks : 36

Note : The candidates are required to attempt two questions each from Section A and B carrying 5½ marks each and the entire Section C consisting of 10 short answer type questions carrying 14 marks.

Section - A

1. (a) Let V is a vector space of all 3 ×3 matrices over R. Find the dimension of V.
   (b) Prove that intersection of two vector sub spaces of a vector space is again a vector sub space. Is the same true about union of two sub space ? Justify. 2.5,3
2. (a) If V is a vector space over F and S, T are subsets of V. Then prove that $L(S \cup T) = L(S) + L(T)$.
   (b) Prove that a vector space V(F) will be a direct sum of two sub spaces W_1 and W_2 if and only if (i) $V = W_1 + W_2$ and (ii) $W_1 \cap W_2 = \{0\}$. 2.5, 3
3. (A) Let V be a vector space over the field F. Prove that the sets S of non zero vectors $v_1, v_2, v_3, \ldots v_n \in V$ is linearly dependent if and oly if one of these vectors say v_k, $2 \leq k \leq n$, can be expressed as a linear combination of vectors preceding it in the set S.
   (b) If V(F) is a finite dimensional vector space and W is a sub space of V, prove that dim (V/W) = dim (V) – dim(W). 2.5,3
4. (a) Find basis and dimension of the sub space W of R^4, generated by the vectors : (1, –2, 5, –3), (2, 3, 1, –4), (3, 8, –3, –5).
   (b) If V(F) is a finite dimension vector space. Prove that any two basis of V will have same number of elements. 2.5,3

Section - B

5. Prove that every n–dimensional vector space over the field F is isomorphic to the space F^n. 5.5
6. State and prove Rank Nullity theorem. 5.5
7. (a) Find characteristic polynomial of the linear transformation $T : R^2 \to R^2$ defined as $T(x, y) = (x - y, x + y)$ and verify Cayley Hamilton theorem.
   (b) If the matrix, of a linear transformation $T : R^3 \to R^3$ relative to the usual basis is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.
   Find the matrix of T relative to the basis B = {(0, 1, –1), (–1, 1, 0), (1, –1, 1)}. 2.5, 3
8. (a) If T : V → W is an isomorphism of V onto W, prove that T maps a basis of V to a basis of W.
   (b) Let T_1, T_2 be two linear operators on a vector space V(F) of dimension n. A and B = $\{v_1, v_2, v_3, \ldots, v_n\}$, is an ordered basis of V. Prove that $[T_1 + T_2. B] + [T_2 ; B]$. 2.5, 3

Section - C

9. (i) Show that there is no non singular linear transformation from R^5 to R^4.
   (ii) Does the set of all polynomials over real numbers with constant term an even integer, form a Vector space ?
   (iii) Examine whether (1, –3, 5) belong to the linear span generated by S = {(1, 2, 1), (1, 1, –1), (4, 5, –2)} over R, or not ?
   (iv) If $V_1, V_2, V_3, \ldots, V_n$, are a linearly independent (L.I.) n × 1 column vectors and A is a n × n non singular matrix, prove that $AV_1, AV_2, AV_3, \ldots, AV_n$ are alos L.I.

(v) Check the Linear independence of the vectors.
$(1, 2, -3), (1, -3, 2), (2, -1, 5) \in V_3(R)$.

(vi) Let U, V, W be vector spaces over the same field F and $T_1 : U \to V$ and $T2 : V \to W$ be linear transformations. Prove that $T_2T_1 : U \to W$ is also a linear transformation. $1\times6=6$

(vii) Find a Linear transformation $T : R^3 \to R^3$, whose range space is generated by (1, 2, 3) and (4, 5, 6).

(viii) Let λ be an Eigen value of an invertible operator T on a vector space V(F). Prove that λ^{-1} is an Eigen value of T^{-1}.

(ix) Define transition matrix. If $B_1 = \{(-1, 0), (0, 1)\}$ and $B2 = \{(1, -2), (2, 3)\}$ are two Basis of R^2. Find transition matrix from B_2 to B_1.

(x) Define Kernel of a Homomorphism f from a vector space V to a vector space W over F. Also prove that Kernel of f is a sub space of V. $2\times4=8$