THEORY OF NUMBERS – III (i)

Time Allowed : Three Hours] [Maximum Marks : 100

Note : Attempt *one* quesiton each from Section A, B, C and D carrying 20 marks each and the entire Section E consisting of 8 short answer type questions carrying 20 marks in all.

Section : A

1. (a) Prove that every composite number can be represented as product of prime factors in one way only. {Different orders of the same factors will not be consisdered as different representations.}

   (b) Let $n > 1$ and k be any positive integer. Prove that $(n-1)^2/(n^k - 1)$ if and only if $(n-1)/k$.

   (c) Prove that no polynomial $f(x)$ of degree > 1 with integral coefficient can represent a prime for each positive integer x. 7,7,6

2. (a) If p is a prime number, then $(p-1)! \equiv -1 \pmod{p}$. What can be said about the

converse. Justify.

(b) Find the least residue of 7^{973} (mod 72).

(c) If a is selected at random from {1, 2, ... 14} and b is selected from {1, 2, ... 15}, what is the probability that $ax \equiv b \pmod{15}$ has at least one solution ? 7,7,6

Section : B

3. (a) A basket contains a number of eggs. If the eggs are taken out 4, 5, 7 at a time, those left in the basket at the end are 3, 4, 2 respectively. How many eggs are there in the basket if it is full to its maximum capacity of 500 eggs ?

(b) Let the order of a (mod n) be d, then prove that d divides $\phi(n)$.

(c) Let $(a, m) = 1$, and let g be a primitive root of m, then there exists a unique integer h such that $g^h \equiv a \pmod{m}$, $0 \leq h < \phi(m)$. 7,7,6

4. (a) Let $(m, n) = 1$, then prove that $\phi(mn) = \phi(m)\phi(n)$.

(b) If $d_1, d_2, \ldots\ldots d_k$ are all divisors of n such that their sum if D, find the sum of the reciprocals of these divisors.

(c) Let p be an odd prime. The congruence $x^2 \equiv a \pmod{p}$, $(a, p) = 1$ has a solution if and only if $\frac{p-1}{a^2} \equiv 1 \pmod{p}$. 7,7,6

Section : C

5. (a) Let (x_0, y_0, z_0) be a primitive solution of $x^2 + y^2 = z^2$, then one of the integers x_0, y_0, z_0 is divisible by 5.

(b) Solve $x^2 + y^2 = 85^2$.

(c) Find the suceeding fraction of $\frac{4}{9}$ in F_{20}. 7,7,6

6. (a) Given any irrational number ξ, there exists infinitely many rational numbers $\frac{h}{k}$ such that $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}\,k^2}$.

(b) Let $\frac{a}{b}$ and $\frac{c}{d}$ be two fractions such that $ad - bc = \pm 1$, then every fraction which lies between them has denominator $\geq b + d$. 12,8

Seciton : D

7. (a) Solve $x^2 + y^2 = 317$.

(b) Prove that no prime of the form $4k + 3$ is representable as sum of two squares.

(c) Prove that forms of negative discriminant are definite and forms of positive discriminant are indefinite. 7,7,6

8. (a) Define the arithmetical functio $\theta(x)$. Prove that if n is any positive integer then $\theta(2n + 1) - \theta(n + 1) < 2 \log 2$.

(b) Define Mangolf's function $\Lambda(n)$. Prove that if $n \geq 1$, then $\sum_{d/n} \Lambda(d) = \log n$. 10,10

Section : E

9. Do as directed :

(i) Find the quadratic residues of 25. 2

(ii) If α and β are indices of a(modn). Then. $\alpha \equiv \beta(\mathrm{mod}(\phi(n)))$. 2

(iii) If $b > 0$ then prove that $a = \left[\frac{a}{b}\right] b + f$ for some real f, $0 \leq f < b$, where [.] is greatest ingeger function. 2

(iv) Let $(a, b) = 1$. If a and b both divide n then prove that ab divides n. 2

(v) State Minkowski's Convex body theorem. 3

(vi) Find positive itegers a and b such that they satisfy the equation $(a, b) = 10$ and $[a, b] = 100$. 3

(vii) Solve $9x \equiv 12 \pmod{15}$. 3

(viii) State Euler's summation formula. 3