ALGEBRA – II

Paper - III
Semester-VI

Time Allowed : 3 Hours] [Maximum Marks : 40

Note : Attempt *five* questions in all, selecting *two* question from each Section A and B compulsory question of Section C.

Section - A

1. Let V be a vector space of all functions from R to R. If $U=\{f\in V : f \text{ is even}\}$ and $W=\{g\in V : f \text{ is odd}\}$, then show that
   (i) U and W are subspaces of V.
   (ii) $V = U \oplus W$. 8

2. (a) Show that the set of all real valued continuous functions y = f(x) satisfying the differential equation
   $$y''' + 5y'' + 11y' + 6y = 0$$
   is a vector space over R. Find also the basis of the vector space. 4
   (b) Under what condition on scalar τ do the vectors $(0, 1, \tau)$, $(\tau, 0, 1)$ and $(\tau, 1, 1+\tau)$ forms a basis of C^3.

3. (a) If x, y and z are vectors in vector space over F such that x + y + z = 0, then show that x and y span the same subspace as y and z. 4
   (b) Show that the vectors (1, 1, 2, 4), (2, –1, –5, 2). (1, –1, –4, 0) and (2, 1, 1, 6) are linearly dependent in R^4. 4

4. Let W be a subspace of C^3 over C spanned by $v_1 = (1, 0, i)$, $v_2 = (i, 0, 1)$. Prove that (i) v_1, v_2 froms a basis of W (ii) $u_1 = (1 + i, 0, 1 + i)$, $u_2 = (1 - i, 0, i - 1)$ forms a basis of W. Also find then matrix of ordered basis $B' = (u_1, u_2)$ relative to the ordered basis $B = \{v_1, v_2\}$. 8

Section - B

5. (a) State and prove Sylvester's law of nullity. 4
   (b) Prove that a linear transformation T is one-one if and only if Ker(T) = (0). 4

6. (a) Show that a linear transformation $T : V \rightarrow W$ is non-singular iff T carries each linearly independent subspace of V onto linearly independent subspace of W. 4
   (b) Let T be a linear operator on V and Rank (T^2) = Rank (T). Then show that the Range (T) $\cap$ Ker(T) = (0). 4

7. (a) Show that the minimal polynomial of
   $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 3 \end{bmatrix}$$
   is $x^4 + 3x^3 + x - 1 = 0$. 4
   (b) Let T be a linear operator on n dimensional space V. Then show that the characteristic and minimal polynomials for T have the same roots. 4

8. Let A be an $n \times n$ matrix over F. Show that A is invertible if and only if column of A are linearly independent over F. 8

Section - C
(Compulsory Question)

9. (a) Show that the union of two subspaces of a vector space may not be subspace.
   (b) Show that $\{1, \sqrt{2}\}$ is linearly independent in R over Q.
   (c) Find two linear transformations T and U on R^2 such that TU = 0 but UT $\neq$ 0.
   (d) Find the value fo k so that the vectors (1, –1, 3), (1, 2, –2) and (k, 0, 1) are linearly dependent.

(e) Show that the Kernal of a linear transformation $T : V \rightarrow W$ is subspace of V.

(f) Check whether a mapping $T : R^2 \rightarrow R$ defined by $T(x, y, z) = x^2 + y^2 + z^2$ is a linear transformation ?

(g) Extent the set $S = \{1, 1, 1\}$ as a basis of R^3.

(h) Find the co-ordinate vectors if v in R^3 relative to the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.

(i) Show that E_1E_2 is a projection if $E_1E_2 = E_2E_1$ where E_1 and E_2 are projections.

(j) Show that the minimal polynomial of a linear operator T divides its characteristic polynomial.

(10×8=8)