ALGEBRA–II

Paper-III : Semester-VI

Time Allowed : Three Hours **Maximum Marks : 40**

Note : The candidates are required to attempt two questions each from Section A and B carrying 7½ marks each and the entire Section C consisting of 10 short answer type questions carrying 1 marks each.

SECTION-A

1. If U and W are subspaces of a finite dimensional vector space over F, then prove that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. 7½

2. (a) Show that the set of all real valued continuous functions $y = f(x)$ satisfying the differential equation $y''' + 6y'' + 11y' + 6y = 0$ is a vector space over R. Find also the basis of the vector space. 4

   (b) Under what condition on scalar τ do the vectors $(1, 1, 1)$, $(1, \tau, \tau^2)$ and $(1, -\tau, \tau^2)$ forms a basis of C^3? 3, 5

3. (a) If x, y and z are vectors in vector space over F such that $x + y + z = 0$, then show that x and y span the same subspace as y and z. 3½

   (b) Let V be a vector space over an infinite field. Prove that V can not be written as set theoretic union of a finite number of proper subspaces. 4

4. (a) Find the matrices of ordered basis (f_1, f_2, f_3) relative to standard basis (e_1, e_2, e_3) of vector space R^3 if $f_1 = (1, \cos x, \sin x)$, $f_2 = (1, 0, 0)$ and $f_3 = (1, -\sin x, \cos x)$. 3½

   (b) Prove that every subspace of a finite dimensional vector space has a complement. 4

SECTION–B

5. (a) Show that a linear transformation $T : V \to V$ is singular if and only if its minimal polynomial has nonzero constant term. 4

   (b) Let T be a linear operator on V and $\text{Rank}(T^2) = \text{Rank}(T)$, then show that the Range $(T) \cap \text{Ker}(T) = (0)$. 3½

6. (a) State and prove Sylvester's law of nullity. 4

   (b) Let $T : R^4 \to R^3$ be a linear transformation defined by $T(x, y, z, t) = (x - y + z + t, 2x - 2y + 3z + 4t, 3x - 3y + 4z + 5t)$, then find a basis and dimension of the image of T. 3½

7. (a) Let T be a linear operator on n dimensional space V, then show that the characteristic and minimal polynomials for T have the same roots. 4

   (b) Prove that a linear transformation T is one-one if and only if $\text{Ker}(T) = (0)$. 3½

8. Let U and V be two vector spaces over the same field F dimensions n and m respectively, then show that homomorphism from U to V is isomorphic to the vector space of all m × n matrices over F. 7½

SECTION–C

9. (i) Show that any n + 1 member of vector space V of dimension n are linearly dependent.

(ii) Show that $\{1, \sqrt{2}\}$ is linearly independent in R over Q.

(iii) Define the Quotient space with suitable example.

(iv) Find the value of k so that the vectors (1, –1, 3), (1, 2, –2) and (k, 0, 1) are linearly dependent.

(v) Determine the complement of the subspace of V generated by {(1, 1, 0), (0, 1, 0)}.

(vi) Check whether a mapping $T : R^3 \rightarrow R$ defined by $T(x, y, z) = x^2 + y^2 + z^2$ is a linear transformation.

(vii) Extend the set S = {(1, 1, 1)} as a basis of R^3.

(viii) Show that E_1E_2 is a projection if $E_1E_2 = E_2E_1$ where E_1 and E_2 are projections.

(ix) Give an example of a Linear transformation T on R^2 such that $T^2 = 0$ but $T \neq 0$.

(x) Find the rank of the linear transformation $T : R^2 \rightarrow R^3$ such that $T(x, y) = (x, x + y, y)$.