NUMBER THEORY - II (I)

Time : Three Hours] [Maximum Marks : 40

Note : Attempt *five* questions selecting *two* questions each from Section A and B and the compulsory question of Section C.

Section - A

1. (a) Show that the continued fraction which represents a quadratic surd is periodic. 4

   (b) Show that there are no positive integral solutions of $x^4 + y^4 = z^2$. 4

2. (a) Show that

$$\prod_{n=0}^{\infty}\left\{\left(1-x^{5n+2}\right)\left(1-x^{5n+3}\right)\left(1-x^{5n+5}\right)\right\}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(5n+1)}$$ 4

(b) Show that for any real number α, there exists a rational number $\frac{p}{q}$ such that $\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^2}$. 4

3. (a) Show that there exists integers x and y not both 0, for which $\xi^2+\eta^2 \le \left(\frac{3}{4}\right)^{1/2} |\Delta|$. 4

(b) State and prove Minkowski's theorem in Geometry of Numbers. 4

4. (a) State and prove Euler's recursion formula for p(n), where p(n) is the number of partitions of n. 4

(b) Show that he number of partitions of n into distinct parts is equal to number of partitions of n into odd parts by means of generating functions. 4

Section - B

5. (a) Show that the average order of $\sigma_1(n)$ is $\pi^2 n/12$. 4

(b) Prove that if a prime, $p \equiv 1 \pmod 4$ hen we can write it as the sum of two integer squares, i.e., $p=\lambda_1^2+\lambda_2^2$, for some $\lambda_1, \lambda_2 \in Z$. 4

6. (a) Show that the following relations are logically equivalent :

$$\lim_{x\to\infty}\frac{\theta(x)}{x}=1 \text{ and } \lim_{x\to\infty}\frac{\psi(x)}{x}=1.$$ 4

(b) Let f (n) = $\left[\sqrt{n}\right]-\left[\sqrt{(n-1)}\right]$. Prove that f is multiplicative but not completely multiplicative. 4

7. (a) Prove that

$$\sum_{n\le x}\frac{d(n)}{n}=\frac{1}{2}\log^2 x+2C\log x+O(1),$$ C is Euler's constant. 4

(b) State and prove Hermite's theorem on minima of positive definite quadratic forms. 4

8. (a) State and prove Abel's Identity. 4

(b) Show that

$$\psi(x)=\sum_{m\le\log_2 x}\theta\left(x^{1/m}\right)$$

where $\psi(x)$ and $\theta(x)$ are Chebyshev's functions. 4

Section - C

9. (a) Define average order of an arithmetic function. Give average order of $\mu(n)$. (Do not prove). 1

(b) State Jacobi's Triple Product Identity. 1

(c) Give generating function for partitions into even and unequal parts. 1

(d) Give the recurrence relation of Pentagonal numbers. 1

(e) State Hermite Theorem on quadratic forms. 1

(f) Exhibit a solution of $x^2 - 13y^2 = 1$. 1

(g) Define Binary quadratic forms with example. ½

(h) Calculate the highest power of 10 that divides 1000 ! ½

(i) For the Pell number, derive the relation, where $n \ge 1$; $p_n + p_{n-1} = q_n$. ½

(j) Is the function $f(n)=\frac{3n+17}{2n-1}$ is O(1), (where O-Big Oh notation) ? Justify your answer. ½