MATHEMATICS Paper–I

(Advanced Calculus –I)

Time Allowed : 3 Hours **Max. Marks : 30**

Note : Attempt five questions in all, selecting at least two from each Unit. All questions carry equal marks.

UNIT–I

1. (a) Let a function f be defined by

$$f(x, y) = \frac{x^3 - y^3}{x^3 + y^3}, (x, y) \neq (0, 0)$$

Show that the two iterated limits $\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\}$ and $\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\}$ exist by the simultaneous limit $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

(b) Discuss the continuity of the function

$$f(x,y)=\begin{cases} x\,y \sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$ at the point (0, 0).

2. (a) If $x^x y^y z^z = c$, show that $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$, when $x=y=z$.

(b) If $u = f(r)$, where $r=\sqrt{x^2+y^2}$, prove that :

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

3. (a) Show that $f(x, y) = \sin x + \cos y$ is differentiable at every point of R^2.

(b) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$, where

$$f(x,y)=\begin{cases} x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right) & \text{if } xy\neq 0 \\ 0 & \text{if } xy=0 \end{cases}$$

4. (a) For the function $f=\frac{y}{x^2+y^2}$, find the value of directional derivative making an angle 30° with the positive x-axis at the point (0, 1).

(b) Prove that curl curl $\vec{A}$ = grad div $\vec{A}-\nabla^2\vec{A}$.

UNIT–II

5. (a) State and prove Euler's theorem on homogeneous functions of two variables.

(b) State Taylor's theorem for function of two variables and use it to expand $x^2y-3y+3$ in powers of $x+1$ and $y-2$.

6. (a) If u, v, w are the roots of the equation

$$(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$$

Prove that : $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

(b) Show that the functions

$u = x + y - z,\ v = x - y + z,\ w = x^2 + y^2 + z^2 - 2yz$

are not independent of one another. Also find the relation between u, v and w.

7. (a) Find the extreme values of the function :

$f(x, y) = (x - y)^4 + (y - 1)^4.$

(b) Show that the volume of the largest parallelopiped that can be inscribed in the sphere.

$x^2 + y^2 + z^2 = a^2$ is $\frac{8a^2}{3\sqrt{3}}$.

8. (a) Prove that the envelope of the circles which pass through the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and have their centres upon its circumference is the curve $x^2 + y^2 = 4(a^2x^2 + b^2y^2)$.

(b) If ρ_1 and ρ_2 are the radii of curvature at the corresponding points of a cycloid and its evolute, prove that :

$\rho_1^2 + \rho_2^2 = \text{constant}.$