MATHEMATICS Paper–I

(Advanced Calculus – II)

Time Allowed : 3 Hours **Maximum Marks : 30**

Note : Attempt five questions in all, selecting at least two from each unit. All questions carry equal marks.

UNIT – I

1. (a) Prove that the sequence $\{a_n\}$, where $a_n = \dfrac{2n-1}{3n+5}$ converges to $\dfrac{2}{3}$.

(b) If s_1 and s_2 are positive and $s_{n+1} = \sqrt{s_n s_{n-1}}$, prove that sequences $s_1, s_3, s_5, \ldots; s_2, s_4, s_6, \ldots$ are the one increasing and the other decreasing and their common limit is $(s_1 s_2^2)^{1/3}$.

2. (a) Show that the sequence $\{a_n\}$, where $a_1 = 1$ and $a_{n+1} = \sqrt{6+a_n}$ converges to 3.

(b) State and prove Cauchy's second theorem on limits.

3. (a) Show that the sequence $\{a_n\}$, where : $a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2n-1}$

is not convergent. Prove that $\{a_n\}$ diverges to ∞.

(b) Show that : $\lim_{n \to \infty} n^{1/n} = 1$

4. (a) Show that the function f defined by : $f(x) = \begin{cases} x, \text{ if } x \text{ is irrational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$

is continuous only at $x = 0$.

(b) Show that $f(x) = \cos x$ is uniformly continuous on $\left[0, \frac{\pi}{2}\right]$.

UNIT–II

5. (a) Show that the alternating series : $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

is convergent.

(b) Discuss the convergence or divergence of the following series :

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$$

6. (a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 1$ converges and its sum lies between $\frac{1}{p-1}$ and $\frac{p}{p-1}$.

(b) Examine the convergence or divergence of $\sum_{n=1}^{\infty} e\sqrt{n}_r n, r > 0$.

7. (a) Discuss the convergence or divergence of the series

$$1 + \frac{1}{2} \cdot \frac{a}{b} + \frac{1.3}{2.4} \cdot \frac{a(a+1)}{b(b+1)} + \frac{1.3.5}{2.4.6} \cdot \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots, a > 0, b > 0$$

(b) Examine the convergence or divergence of the series :

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n, x > 0.$$ converges if β > α and diverges if β ≤ α.

8. (a) Prove that the series $x - \frac{x^3}{3} + \frac{x^4}{5} - \frac{x^7}{7} + \ldots\ldots.$ is convergent for $-1 < x \leq 1$.

(b) Assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the value of $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.