MATHEMATICS: PAPER-I

(Analysis-I)

Time Allowed: Three Hours

Maximum Marks: 65

Note: Attempt five questions in all, selecting at least one question from each Section. All questions carry equal marks.

SECTION-A

- 1. (a) Prove that the set $\{....2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3, ...\}$ is countable.
 - (b) If $0 \le a \le b$, show that :

$$\left| \int_{a}^{b} \frac{\sin x}{x} dx \right| \le \frac{2}{a}$$

- 2. (a) If f is continous on [a, b], then $f \in R(x)$ on [a, b].
- (b) Give an example of a bounded function f defined on a closed interval such that |f| is R-integrable but f is not.
- 3. (a) If f is R-integrable on [a, b] and k is any real number, then k_f is also R-integrable and $\int_a^b (kf)(x)dx = k \int_a^b f(x)dx$
 - (b) Show that:

$$\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{1}{(a+b)^m a^n} \beta(m,n)$$

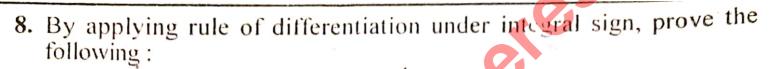
- 4. (a) Prove that : $\Gamma(m)\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$
 - (b) Show that $\int_0^\infty \frac{x}{1+x^6} dx = \frac{\pi}{3\sqrt{3}}$

SECTION-B

- 5. (a) Discuss the convergence of $\int_0^\infty \frac{x \tan^{-1}}{(1+x^4)^{1/3}} dx$.
 - (b) Discuss the convergence of $\int_a^b \frac{dx}{(x-a)\sqrt{b-x}}$.
- **6.** (a) If $\phi(x)$ is bounded and monotonic in [a, ∞] and $\int_a^{\infty} f(x)dx$ is convergent at ∞ , then prove that $\int_a^{\infty} f(x)\phi(x)dx$ is convergent at ∞ .
 - (b) Show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx$ is convergent. Also using

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}, \text{ show that } \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

- 7. (a) Show that $\int_0^{\pi/2} \sin x \cdot \log \sin x \, dx$ is convergent and its value is $\log \frac{2}{e}$.
 - (b) Show that $\int_0^\infty \frac{\sin ax \sin bx}{x} dx$ converges to $\frac{1}{2} \log \left(\frac{a+b}{a-b} \right)$ where a+b are positive reals.



(a)
$$\int_0^{\pi/2} \log \left(\frac{a + b \sin \theta}{a - b \sin \theta} \right) \csc \theta \, d\theta = \pi \sin^{-1} \frac{b}{a}.$$

(b)
$$\int_0^{\pi/2} \left(\frac{\log(1 + y \sin^2 x)}{\sin^2 x} \right) dx = \pi \left[\sqrt{1 + y} - 1 \right].$$