MATHEMATICS : PAPER—I

(Analysis-I)

Time Allowed : Three Hours **Maximum Marks : 65**

Note : Attempt **five** questions in all, selecting at least **one** question from each Section. All questions carry equal marks.

SECTION – A

1. (a) Prove that the set $\{ \ldots 2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3, \ldots\}$ is countable.

   (b) If $0 < a < b$, show that :

$$\left|\int_a^b \frac{\sin x}{x} dx\right| \leq \frac{2}{a}$$

2. (a) If f is continous on [a, b], then $f \in R(x)$ on [a, b].

   (b) Give an example of a bounded function f defined on a closed interval such that $|f|$ is R-integrable but f is not.

3. (a) If f is R-integrable on $[a, b]$ and k is any real number, then k_f is also R-integrable and $\int_a^b (kf)(x)dx = k\int_a^b f(x)\,dx$

   (b) Show that :

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{1}{(a+b)^m a^n} \beta(m,n)$$

4. (a) Prove that : $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

(b) Show that $\int_0^{\infty}\frac{x}{1+x^6}dx=\frac{\pi}{3\sqrt{3}}$

SECTION – B

5. (a) Discuss the convergence of $\int_0^{\infty}\frac{x\tan^{-1}}{(1+x^4)^{1/3}}dx.$

(b) Discuss the convergence of $\int_a^b\frac{dx}{(x-a)\sqrt{b-x}}.$

6. (a) If $\phi(x)$ is bounded and monotonic in [a, ∞] and $\int_a^{\infty}f(x)dx$ is convergent at ∞, then prove that $\int_a^{\infty}f(x)\phi(x)dx$ is convergent at ∞.

(b) Show that $\int_0^{\infty}\frac{\sin^2 x}{x^2}dx$ is convergent. Also using

$\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$, show that $\int_0^{\infty}\frac{\sin^2 x}{x^2}dx=\frac{\pi}{2}$.

7. (a) Show that $\int_0^{\pi/2}\sin x.\log\sin x\,dx$ is convergent and its value is $\log\frac{2}{e}$.

(b) Show that $\int_0^{\infty}\frac{\sin ax\sin bx}{x}dx$ converges to $\frac{1}{2}\log\left(\frac{a+b}{a-b}\right)$ where $a+b$ are positive reals.

8. By applying rule of differentiation under integral sign, prove the following :

(a) $\int_0^{\pi/2} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\operatorname{cosec}\theta\, d\theta = \pi \sin^{-1}\frac{b}{a}$.

(b) $\int_0^{\pi/2} \left(\frac{\log(1+y\sin^2 x)}{\sin^2 x}\right) dx = \pi\left[\sqrt{1+y}-1\right]$.