MATHEMATICS : PAPER II

(Modern Algebra)

Time Allowed : Three Hours **Maximum Marks : 30**

Note : Attempt **five** questions in all, selecting at least **two** questions from each Section. All questions carry equal marks.

SECTION – A

1. (a) Let G be a finite group and $a \in G$ be an element or order n. Then $a^m = e$ iff n is a divisor of m.

   (b) If H is a sub-group of G of index 2 in G, then prove that H is normal subgroup of G.

2. (a) Let N be a normal sub-group of a group G. Show that G/N is abelian iff for all $x, y \in G$, $xyx^{-1}y^{-1} \in N$.

   (b) Let H and K be two subgroups of a group G. Show that :

   $(H \cap K)C = H_C \cap K_C$ for all $C \in G$.

3. (a) Let G and G′ be two groups and $f : G \to G'$ homomorphism of G onto G′. Then prove that $\frac{G}{H} \cong G'$ where H is the kernel of f.

   (b) If $f : G \to G$ defined by $f(x) = x^3 \ \forall \ x \in G$ is an automorphism, show that G is abelian.

4. (a) If $O(G) = p^2$, where p is prime number then prove that G is an abelian group.

   (b) Prove that A_4 has no subgroup of order six. A_4 is alternative group of order 4.

SECTION – B

5. (a) If in a ring R, $x^3 = x$ for all $x \in R$, then show that R is commutative.

   (b) Let R be a finite ring without zero divisors and $O(R) > 1$. Then show that R is a division ring.

6. (a) Prove that intersection of two left (or right) ideals of a ring is a left (or right) ideal.

(b) Prove that a commutative ring with unity is simple iff R is a field.

7. (a) Show that any ideal of $\mathbb{Z}$ is maximal iff it is generated by some prime element.

(b) Show that the mapping $f : \mathbb{Z} \to M_2(\mathbb{Z})$, defined by : $f(r) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ is a homomorphism of rings, find ker f. Is the mapping an isomorphism ?

8. (a) Let R be a commutative ring with identity and $f(x).g(x) \in R[x]$. Then $\deg.[f(x)\,g(x)] \leq \deg.\, f(x) + \deg g(x)$.

(b) For any ring R, show that $R[x]/<x> \cong R$.