MATHEMATICS : PAPER III

(Probability Theory)

Time Allowed : Three Hours **Maximum Marks : 30**

Note : Attempt **five** questions in all, selecting at least **two** questions from each Section. All questions carry equal marks.

SECTION – A

1. (a) A typical PIN (Personal Identification Number) is a sequence of any *four* symbols chosen from 26 letters in the alphabet and the ten digits. If all PINs are equally likely, find the probability that a randomly chosen PIN contains a repeated symbol.

(b) State and prove Bayes' theorem.

2. (a) A bowl contain 10 balls of same size and shape out of which one of the balls is red. Balls are drawn one by one at random and without replacement until the red ball is drawn. Find the p. m. f. and c. d. f. of random variable X, the number of trials needed to draw the red chip.

(b) Let X be a continuous random variable having p. d. f. :

$$f(x) = \begin{cases} 6x(1-x) & , \quad 0 \le x \le 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find a number m such that $P(X < m) = P(X > m)$.

3. (a) Let a random variable X of continous types has a probability density function $f(x)$, whose graph is symmetric with respect to $x = c$. If the mean value of X exists, then show that E (X) = c.

(b) Let X be a continuous random variable having p . d. f. :

$$f(x) = \begin{cases} \frac{3}{4}x(2-x) & , \quad 0 \le x \le 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find measure of skewness and kurtosis of the distribution.

4. (a) If a fair coin is tossed at random five indepedent times, find the conditional probability of five heads relative to the hypothesis that there are at least four heads.

(b) If for a Poisson random variable X, $E(X^2) = 20$, find E (X).

SECTION – B

5. (a) Let X be uniformly distributed over $[-\alpha, \alpha]$, where $\alpha > 0$. Find α so that :

(i) $P(X > 1) = \frac{1}{3}$

(ii) $P\left(X < \frac{1}{2}\right) = 0.8$

(iii) $P(|X| < 1) = P(|X| > 1)$

(b) Find coefficients of skewness and kurtosis of an exponential distribution and describe nature of the distribution.

6. (a) If X has gamma distribution with $\alpha = \frac{r}{2}$, $r \in N$ and $\beta > 0$, then show that $Y = \frac{2X}{\beta}$ is $\chi^2(r)$.

(b) There are six hundred mathematics students in the graduate classes of a university and the probability for any student to need a copy of a particular book from the university library on any day is 0.05. How many copies of the book should be kept in the university library so that probability may be greater than 0.90 that none of the students needing a copy from the library has to come back disappointed ?

7. (a) Let two dimensional continuous random variable (X, Y) has joint probability density function given by :

$$f(x,y)=\begin{cases}6x^2y & , \quad 0<x<1, 0<y<1 \\ 0 & , \quad \text{elsewhere}\end{cases}$$

(i) Verify that $\int_0^1\int_0^1 f(x,y)\,dx\,dy=1$.

(ii) Find $P(0<X<\frac{3}{4}, \frac{1}{3}<Y<2)$.

(iii) Find $P(X+Y<1)$.

(iv) Find $P(X>Y)$.

(v) Find $P(X<1 \mid Y<2)$.

(b) Let $f(x,y)=\begin{cases}8xy & , \quad 0<x<y<1 \\ 0 & , \quad \text{elsewhere}\end{cases}$

Find :

(i) $E(Y \mid X=x)$

(ii) $E(X \mid Y=y)$

(iii) $Var(Y \mid X=x)$

(iv) $Var(X \mid Y=y)$

8. (a) Let the variables X and Y be connected by equation $aX+bY+c=0$. Prove that the coefficient of correlation between X and Y is -1 if a and b are of same sign and $+1$ if a and b are of opposite signs.

(b) In a certain population of married couples the height X of the husband and the height Y of the wife has a bivariate normal distribution with parameters $\mu_X=5.8$ feet, $\mu_Y=5.3$ feet, $\sigma_x=\sigma_y=0.2$ feet and $r=0.6$. If height of the husband is 6.3 feet, find the probability that his wife has a height between 5.28 and 5.92 feet.