MATHEMATICS : PAPER—I

(Analysis -II)

Time Allowed : Three Hours **Maximum Marks : 30**

Note : Attempt any *five* questions in all, selecting at least *two* from each Unit. All questions carry equal marks.

UNIT–I

1. (a) Let $A=\{(x, y): -1 \leq x \leq 1, -1 \leq y \leq 1\}$ and $f: A \rightarrow R$ be defined by :

$$f(x, y) = \begin{cases} y & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that $\int_{-1}^{1}\left(\int_{-1}^{1} f(x, y)\, dy\right) dx$ exists and the other repeated integral is not defined.

(b) Change the order of integration and hence evaluate

$$\int_0^a \int_x^{\frac{a^2}{x}} (x+y)\, dx dy.$$

2. (a) Find the area of the region bounded between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$.

(b) Evaluate $\iiint xyz(x^2+y^2+z^2)\,dx\,dy\,dz$ over $x^2+y^2+z^2=a^2$ in positive octant.

3. (a) Show that $\iiint (x+y+z)^9\,dx\,dy\,dz$ over the region defined by

$$x \geq 0, y \geq 0, z \geq 0, \quad x+y+z=1 \text{ is } \frac{1}{24}.$$

(b) Show that $\vec{F}=(2xy+z^3)\hat{i}+x^2\hat{j}+3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. Also find the work done in moving an object in this field from (1, –2, 1) to (3, 1, 4).

4. (a) State and prove Gauss's divergence theorem.

(b) Verify Stokes' theorem for the vector point function $\vec{F}=z\hat{i}+x\hat{j}+y\hat{k}$, where curve is the unit circle in the XY plane bounding the semi-sphere $z=\sqrt{1-x^2-y^2}$.

UNIT–II

5. (a) Show that sequence $\{f_n(x)\}$ where $f_n(x)=\dfrac{n}{x+n}$ is uniformly convergent on $[0,k]$, where k is any positive real number but is not uniformly convergent on $[0, \infty]$.

(b) Show that the series $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n}$ converges uniformly in $(0,2\pi)$.

6. (a) Test for uniform convergence and term by term integration of the series :

$$\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}, 0 \leq x \leq 1$$

(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^4 x^2}$ is uniformly convergent for all x and it can be differentiated term by term.

7. (a) Prove that the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$ converges for $-1 < x \leq 1$.

(b) Prove that :

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6....2n} \frac{x^{2n+1}}{2n+1} \quad \forall x \in [-1,1].$$

Hence deduce that $\frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + ...$

8. (a) Find a series of sines and cosines of multiples of x which represents $x + x^2$ in $(-\pi, \pi)$. Hence show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + ...$

(b) If $f(x) = \begin{cases} \frac{\pi}{3}; & 0 \leq x \leq \frac{\pi}{3} \\ 0; & \frac{\pi}{3} \leq x \leq \frac{2\pi}{3} \\ \frac{-\pi}{3}; & \frac{2\pi}{3} \leq x \leq \pi \end{cases}$

then show that :

$$f(x) = \frac{2}{\sqrt{3}} \left[\cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - .... \right]$$

Hence deduce that $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + ...$