

MATHEMATICS : PAPER II

(Linear Algebra)

Time Allowed : Three Hours

Maximum Marks : 30

Note : Attempt any *five* questions in all, selecting at least *two* from each.

UNIT-I

1. (a) Let V be the set of positive real numbers. Let vector addition and scalar multiplication be defined as :

$$x + y = xy \text{ and } \alpha x = x^\alpha \text{ for all } x, y \in V \text{ and } \alpha \in \mathbf{R}.$$

Prove that $V_{\mathbf{R}}$ is a vector space.

- (b) Show that the only non-trivial subspaces of \mathbf{R}^2 over field \mathbf{R} are the lines through origin.

2. (a) If $V_{\mathbf{F}}$ is a finitely generated vector space, then prove that all the bases of V have the same number of elements.

- (b) Let W be a subspace of $F_3[x]$ generated by polynomials :

$$f_1 = x^3 - 2x^2 + 4x + 1,$$

$$f_2 = 2x^3 - 3x^2 + 9x - 1,$$

$$f_3 = x^3 + 6x - 5$$

and

$$f_4 = 2x^3 - 5x^2 + 7x + 5.$$

Find a basis and dimension of W .

3. (a) Find a basis and dimension of the solution space of systems of homogeneous linear equations :

$$x + 2y - 4z + 3s - t = 0$$

$$x + 2y - 2z + 2s + t = 0$$

$$2x + 4y - 2z + 3s + t = 0$$

- (b) State and prove Rank Nullity theorem.

4. (a) Find a linear transformation $T : \mathbf{R}^4 \rightarrow \mathbf{R}^3$ whose kernel is spanned by $(0, 1, 2, 3)$ and $(-1, 2, 3, 0)$.

(b) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined as :

$$T(x, y, z) = (2x, 4x - y, 2x + 3y - z).$$

Prove that T is invertible and find T^{-1} .

UNIT-II

5. (a) Let T be a linear operator on \mathbb{R}^3 defined by :

$$T(x, y, z) = (2x - 3y + 4z, 5x - y + 2z, 4x + 7y).$$

(i) Find matrix of T w. r. t. standard basis \mathcal{B} of \mathbb{R}^3

(ii) Verify that $[T; \mathcal{B}][v; \mathcal{B}] = [T(v); \mathcal{B}] \quad \forall v \in \mathbb{R}^3$

(b) If the matrix of linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ relative to usual

basis is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$, then find the matrix of T relative to the

basis $\mathcal{B}' = \{(0, 1, -1), (-1, 1, 0), (1, -1, 1)\}$

6. (a) Show that the matrices $A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ and

$B = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$ are similar over \mathbb{C} .

(b) Find eigen values and eigen vectors of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$.

Also verify that geometric multiplicity of an eigen value cannot exceed its algebraic multiplicity.

7. (a) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear operator defined by $T(x, y, z) = (7x + 4y - z, 4x + 7y - z, -4x - 4y + 4z)$. Is T diagonalizable ?

(b) Find the characteristic polynomial of the operator $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + y + z, x + y, x)$ and use Cayley-Hamilton theorem to find T^{-1} .

8. (a) Find minimal polynomial of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence Find A^{-1} .

(b) Prove that characteristic roots of a linear operator $T : V \rightarrow V$ are independent of the choice of basis of V .