MATHEMATICS Paper–II

(Calculus–I)

Time Allowed : 3 Hours **Max. Marks : 30**

Note : Attempt *five* questions in all, selecting at least *two* questions from each Section.

Section-A

1. (a) Solve for x the inequality $\frac{x+2}{n-2} < \frac{4n-1}{2n-3}$.

   (b) Prove that $\left|x - \frac{1}{2}\right| < \frac{1}{3}$ iff $\frac{1}{11} < \frac{1-x}{1+x} < \frac{5}{7}$. 3,3

2. (a) State order completeness property of reals. Does the set of rational numbers possess this property ? Justify your answer.

   (b) Find the least upper bound and greatest lower bound of the set S = $\left\{\frac{2-x}{1-x}; x > 0, x \neq 1\right\}$. 3, 3

3. (a) Is the union of two bounded sets a bounded set ? What do you say about its converse ? Justify your answer.

   (b) If $f(x) = x\left[\frac{1}{x}\right]$, does $\underset{x \to 1/2}{\text{Lt}} f(x)$ exist, explain your answer. 3,3

4. (a) Prove that if a function $f(x)$ is continuous at a point a and $f(a) \neq 0$, then prove that these exists some neighbourhood of a where $f(x)$ possesses the same sign as that if $f(a)$.

   (b) Determine the values of a and b for which :

   $$\underset{n \to 0}{\text{Lt}} \frac{x(1 + a\cosh x) - b\sinh x}{x^3} - 1$$ 3,3

Section-B

5. (a) By using Lagrange's mean vlaue theorem prove that :

   $|\sin x - \sin y| \le |x - y|$ for all $x, y \in R$

   (b) Calculate the approximate value of $\sqrt{24}$ to three decimal places by Taylor's expansion. 3,3

6. (a) Use Maclaurin's theorem to show that :

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots + \frac{(-1)^{n-1}}{n}x^n - \frac{x^n}{(1+\theta x)^n}, 0 < \theta < 1.$$

(b) Use mean value theorem to show that :

$$\frac{x}{6} + \frac{2n-1}{\sqrt{3}} \le \sin^{-1} x \le \frac{x}{6} + \frac{2n-1}{2\sqrt{1-x^2}} \text{ where } \frac{1}{2} \le x < 1.$$ 3,3

7. (a) If $y = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$, show that : $\left(\frac{dy}{dx}\right)^2 = x^2 + a^2$

(b) Prove that $\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}, -1 < x < 1$ and find its derivative also. 3,3

8. (a) Prove that :

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n \lfloor n}{x^{n+1}} \cdot \left[\log x - 1 - \frac{1}{2} - \frac{1}{3} \ldots\ldots\ldots - \frac{1}{n}\right].$$

(b) If $y = \sin m(\sin^{-1}x)$, show that :

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0$$

Hence show that :

$$y_n(0) = \begin{cases} 0, & \text{when } n \text{ is even} \\ m(1^2 - m^2)\ (3^2 - m^2)\ldots\ldots\ldots[(n-2)^2 - m^2] & \end{cases}$$

when n is odd. 3,3