JEE MAINS_8-APRIL-2017
MATHEMATICS

(1) Let $f(x) = 2^{10} \cdot x + 1$ and $g(x) = 3^{10} \cdot x - 1$. If $(fog)(x) = x$, then x is equal to:

(1) $\dfrac{3^{10} - 1}{3^{10} - 2^{-10}}$

(2) $\dfrac{2^{10} - 1}{2^{10} - 3^{-10}}$

(3) $\dfrac{1 - 3^{-10}}{2^{10} - 3^{-10}}$

(4) $\dfrac{1 - 2^{-10}}{3^{10} - 2^{-10}}$

(2) Let $\mathrm{p}(x)$ be a quadratic polynomial such that $\mathrm{p}(0) = 1$. If $\mathrm{p}(x)$ leaves remainder 4 when divided by $x - 1$ and it leaves remainder 6 when divided by $x + 1$; then:

(1) p(2) = 11

(2) p(2) = 19

(3) p(−2) = 19

(4) p(−2) = 11

(3) Let $z \in C$, the set of complex numbers. Then the equation, $2|z + 3i| - |z - i| = 0$ represents:

(1) a circle with radius $\dfrac{8}{3}$

(2) a circle with diameter $\dfrac{10}{3}$.

(3) an ellipse with length of major axis $\dfrac{16}{3}$

(4) an ellipse with length of minor axis $\dfrac{16}{9}$

(4). The number of real values of λ for which the system of linear equations

$$2x+4y-\lambda z=0$$
$$4x+\lambda y+2z=0$$
$$\lambda x+2y+2z=0$$

has infinitely many solutions, is:

(1) 0

(2) 1

(3) 2

(4) 3

5. Let A be any 3×3 invertible matrix. Then which one of the following is not always true?

(1) $\text{adj}(A)=|A|\cdot A^{-1}$

(2) $\text{adj}(\text{adj}(A))=|A|\cdot A$

(3) $\text{adj}(\text{adj}(A))=|A|^2\cdot(\text{adj}(A))^{-1}$

(4) $\text{adj}(\text{adj}(A))=|A|\cdot(\text{adj}(A))^{-1}$

6. If all the words, with or without meaning, are written using the letters of the word QUEEN and are arranged as in English dictionary, then the position of the word QUEEN is:

(1) 44^{th}

(2) 45^{th}

(3) 46^{th}

(4) 47^{th}.

7. If $(27)^{999}$ is divided by 7, then the remainder is:

(1) 1

(2) 2

(3) 3

(4) 6

8. If the arithmetic mean of two numbers a and b, a > b > 0, is five times their geometric mean, then a b

$\frac{a+b}{a-b}$ is equal to:

(1) $\frac{\sqrt{6}}{2}$

(2) $\frac{3\sqrt{2}}{4}$

(3) $\frac{7\sqrt{3}}{12}$

(4) $\frac{5\sqrt{6}}{12}$

9. If the sum of the first n terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}+\sqrt{507}+.....$ is 435 $\sqrt{3}$, then n equals:

(1) 18

(2) 15

(3) 13

(4) 29

10. $\lim_{x\to 3}\frac{\sqrt{3x-3}}{\sqrt{2x-4}-\sqrt{2}}$ is equal to:

(1) $\sqrt{3}$

(2) $\frac{1}{\sqrt{2}}$

(3) $\frac{\sqrt{3}}{2}$

(4) $\frac{1}{2\sqrt{2}}$

11. The tangent at the point (2, −2) to the curve, $x^2y^2 - 2x = 4(1-y)$ (1−y) does not pass through the point:

(1) $\left(4, \frac{1}{3}\right)$

(2) $(8,5)$

(3) $(-4, \ -9)$

(4) (−2, −7)

12. If $y = \left[x + \sqrt{x^2} - 1\right]^{15} + \left[x - \sqrt{x^2} - 1\right]^{15}$, then

$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is equal to :

(1) $125\,y$

(2) $224\,y^2$

(3) $225\,y^2$

(4) $225\,y$

13. If a point P has co-ordinates (0, −2) and Q is any point on the circle, $x^2 + y^2 - 5x - y + 5 = 0$, then the maximum value of $(PQ)^2$ is:

(1) $\frac{25+\sqrt{6}}{2}$

(2) $14+5\sqrt{3}$

(3) $\frac{47+10\sqrt{6}}{2}$

(4) $8+5\sqrt{3}$

14. The integral

$\int \sqrt{1+2\cot x(\operatorname{cosec} x+\cot x)}\, dx$

$\left(0<x<\frac{\pi}{2}\right)$ is equal to :

(where C is a constant of integration)

(1) $4 \log\left(\sin\frac{x}{2}\right)+C$

(2) $2 \log\left(\sin\frac{x}{2}\right)+C$

(3) $2 \log\left(\cos\frac{x}{2}\right)+C$

(4) $4 \log\left(\cos\frac{x}{2}\right)+C$

15. The integral $\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8 \cos 2x}{(\tan x+\cot x)^3} dx$ equals:

(1) $\frac{15}{128}$

(2) $\frac{15}{64}$

(3) $\frac{13}{32}$

(4) $\frac{13}{256}$

16. The area (in sq. units) of the smaller portion enclosed between the curves, $x^2+y^2=4$ and $y^2=3x$, is:

(1) $\frac{1}{2\sqrt{3}}+\frac{\pi}{3}$

(2) $\frac{1}{\sqrt{3}}+\frac{2\pi}{3}$

(3) $\frac{1}{2\sqrt{3}}+\frac{2\pi}{3}$

(4) $\frac{1}{\sqrt{3}}+\frac{4\pi}{3}$

17. The curve satisfying the differential equation, $ydx-\left(x+3y^2\right)dy=0$ and passing through the point (1,1), also passes through the point:

(1) $\left(\frac{1}{4},-\frac{1}{2}\right)$

(2) $\left(-\frac{1}{3},\frac{1}{3}\right)$

(3) $\left(\frac{1}{3},-\frac{1}{3}\right)$

(4) $\left(\frac{1}{4},-\frac{1}{2}\right)$

18. The locus of the point of intersection of the straight lines,

$$tx-2y-3t=0$$

$$x-2ty+3=0 \quad \left(t\in \mathrm{R}\right), is:$$

(1) an ellipse with eccentricity $\frac{2}{\sqrt{5}}$

(2) an ellipse with the length of major axis 6

(3) a hyperbola with eccentricity $\sqrt{5}$

(4) a hyperbola with the length of conjugate axis 3

19. If two parallel chords of a circle, having diameter 4 units, lie on the opposite sides of the centre and subtend angles $\cos^{-1}\left(\frac{1}{7}\right)$ and $\sec^{-1}(7)$ at the centre respectively, then the distance between these chords, is:

(1) $\frac{4}{\sqrt{7}}$

(2) $\frac{8}{\sqrt{7}}$

(3) $\frac{8}{7}$

(4) $\frac{16}{7}$

20. If the common tangents to the parabola, $x = 4y$ and the circle, $x^2 + y^2 = 4$ intersect at the point P, then the distance of P from the origin, is:

(1) $\sqrt{2}+1$

(2) $2\left(3+2\sqrt{2}\right)$

(3) $2\left(\sqrt{2}+1\right)$

(4) $3+2\sqrt{2}$

21. Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then the area (in sq. units) of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse, is:

(1) 8

(2) 32

(3) 80

(4) 40

22. The coordinates of the foot of the perpendicular from the point (1, −2, 1) on the plane containing the lines,

$$\frac{x+1}{6}=\frac{y-1}{7}=\frac{z-3}{8} \text{ and}$$

$$\frac{x-1}{3}=\frac{y-2}{5}=\frac{z-3}{7}, \text{is:}$$

(1) (2, −4, 2)

(2) (−1, 2, −1)

(3) (0, 0, 0)

(4) (1, 1, 1)

23. The line of intersection of the planes

$$\vec{r}.\left(3\hat{i}-\hat{j}+\hat{k}\right)=1 \text{and}$$

$$\vec{r}.\left(\hat{i}+4\hat{j}-2\hat{k}\right)=2, \text{is :}$$

(1) $\dfrac{x-\frac{4}{7}}{-2}=\dfrac{y}{7}=\dfrac{z-\frac{5}{7}}{13}$

(2) $\dfrac{x-\frac{4}{7}}{2}=\dfrac{y}{-7}=\dfrac{z+\frac{5}{7}}{13}$

(3) $\dfrac{x-\frac{6}{13}}{2}=\dfrac{y-\frac{5}{13}}{-7}=\dfrac{z}{-13}$

(4) $\dfrac{x-\frac{6}{13}}{2}=\dfrac{y-\frac{5}{13}}{7}=\dfrac{z}{-13}$

24. The area (in sq. units) of the parallelogram whose diagonals are along the vectors $8\hat{i}-6\hat{j}$ and $3\hat{i}+4\hat{j}-12\hat{k}$, is:

(1) 26

(2) 65

(3) 20

(4) 52

25. The mean age of 25 teachers in a school is 40 years. A teacher retires at the age of 60 years and a new teacher is appointed in his place. If now the mean age of the teachers in this school is 39 years, then the age (in years) of the newly appointed teacher is:

(1) 25

(2) 30

(3) 35

(4) 40

26. Three persons P, Q and R independently try to hit a target. If the probabilities of their hitting the target are $\frac{3}{4}, \frac{1}{2}$ and $\frac{5}{8}$ respectively, then the probability that the target is hit by P or Q but not by R is:

(1) $\frac{21}{64}$

(2) $\frac{9}{64}$

(3) $\frac{15}{64}$

(4) $\frac{39}{64}$

27. An unbiased coin is tossed eight times. The probability of obtaining at least one head and at least one tail is:

(1) $\frac{255}{256}$

(2) $\frac{127}{128}$

(3) $\frac{63}{64}$

(4) $\frac{1}{2}$

28. if $S=\left\{x\in[0,2\pi]:\begin{vmatrix}0 & \cos x & -\sin x\\ \sin x & 0 & \cos x\\ \cos x & \sin x & 0\end{vmatrix}=0\right\}$, then $\sum_{x\in S}\tan\left(\frac{\pi}{3}+x\right)$ is equal to:

(1) $4+2\sqrt{3}$

(2) $-2+\sqrt{3}$

(3) $-2-\sqrt{3}$

(4) $-4-2\sqrt{3}$

29. The value of $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1+x^2}}{\sqrt{1+x^2}-\sqrt{1+x^2}}\right]$, $|x|<\frac{1}{2}, x\neq 0$, is equal to:

(1) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$

(2) $\frac{\pi}{4}+\cos^{-1}x^2$

(3) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$

(4) $\frac{\pi}{4}-\cos^{-1}x^2$

30. The proposition $(\sim p)\vee(p\wedge \sim q)$ is equivalent to:

(1) p ∨ ~q

(2) p → ~q

(3) p ∧ ~q

(4) q → p

PART-2

The composition of f and g is,

$$fog(x) = f(g(x))$$

Thus,

$$f\left(3^{10}x-1\right) = 2^{10}\left(3^{10}x-1\right)+1$$

$$2^{10}\left(3^{10}x-1\right)+1 = x$$

$$2^{10}\cdot 3^{10}x - 2^{10} + 1 = x$$

Solve the above equation:

$$2^{10}\cdot 3^{10}x - x = 2^{10} - 1$$

$$x\left(2^{10}\cdot 3^{10} - 1\right) = 2^{10} - 1$$

$$x = \frac{2^{-10}-1}{2\cdot^{10}3^{10}-1}$$

$$x = \frac{1-2^{-10}}{3^{10}-2^{-10}}$$

2. Let the quadratic polynomial be,

$$p(x) = ax^2 + bx + c \qquad \dots (1)$$

Given $p(0) = 1$ then the value of $c = 1$.

And,

$$p(x) = (x-1)r + 4 \qquad \dots (2)$$

$$p(x) = (x+1)s + 6 \qquad \dots (3)$$

Here r and s are constants

Substitute $x=1$ in equation (2) and compare with equation (1).

$$p(1)=4=a+b+1 \quad \ldots (4)$$

$x=-1$ in equation (3)

$$p(-1)=6=a-b+1 \quad \ldots (5)$$

Solve equations (4) and (5) to find a and b.

$a=4$ and $b=-1$

Then,

$$p(x)=4x^2-x+1$$

Therefore,

$$p(-2)=4(-2)^2-(-2)+1$$
$$p(-2)=19$$

3. Consider the complex equation:

$$z=x+iy$$

Then,

$$2\left|x+(y+3)i\right|-\left|x+(y-1)i\right|=0$$
$$2\left|x+(y+3)i\right|=\left|x+(y-1)i\right|$$
$$2\sqrt{x^2+(y+3)^2}=\sqrt{x^2+(y-1)^2}$$
$$4\left(x^2+(y+3)^2\right)=\left(x^2+(y-1)^2\right)$$

Rearrange the above equation:

$$x^2 + y^2 + \frac{26}{3}y + \frac{35}{3} = 0$$

$$x^2 + \left(y^2 + \frac{13}{3}\right)^2 = \left(\frac{8}{3}\right)^2$$

Thus, the equation represent a circle with radius $\frac{8}{3}$.

4. Consider the linear equation:

$$AX = B$$

Since B = 0, infinitely many solution exits for $D = 0$.

Here D is determinant of A.

$$\begin{vmatrix} 2 & 4 & -\lambda \\ 4 & \lambda & 2 \\ \lambda & 2 & 2 \end{vmatrix} = 0$$

$$2(2\lambda - 4) - 4(8 - 2\lambda) - \lambda(8 - \lambda^2) = 0$$

$$\lambda^3 + 4\lambda - 40 = 0$$

Here the roots of the above equation is,

$$\lambda = 3, -1.5 \pm j3.3$$

And,

The above equation is $f(\lambda)$,

$$f(\lambda) = \lambda^3 + 4\lambda - 40$$

Differentiate the above equation with respect to λ,

$$f'(\lambda) = 3\lambda^2 + 4$$
$$> 0$$

Hence, it has only one real value of λ.

5. Consider the properties of matrices,

$$A^{-1} = \frac{1}{|A|} \cdot adj(A)$$

$$adj(A) = |A| A^{-1} \qquad \ldots (1)$$

Also,

$$|adj(A)| = |A|^{n-1} = |A|^2 \qquad \ldots (2)$$

From equations (1) and (2).

$$adj(adj(A)) = |A|^2 (adj(A))^{-1}$$

So, 1, 2 and 3 are correct option and option 4 occurs only when $|A| = 1$.

Hence, the question is asking for the relation which is not true, so option 4 is correct.

6. According to dictionary, the word QUEEN is formed by the letters E, E, N, Q, and U.

Consider the sequence of words in dictionary starting with,

$$\text{E} \_ \ \_ \ \_ \ \_ = 4!$$
$$= 24$$

$$N\_\ \_\ \_\ \_ = \frac{4!}{2!}$$
$$= 12$$

$$QE\ \_\ \_\ \_ = 3!$$
$$= 6$$

$$QN\ \_\ \_\ \_ = \frac{3!}{2!}$$
$$= 3$$

$$QUEEN = 1$$

Total is,

$$24 + 12 + 6 + 3 + 1 = 46^{\text{th}}$$

7. The expression of the Binomial Theorem is as follows:

$$(1+x)^n = {}^nc_0 x^n (1)^0 + {}^nc_1 x^{n-1} (1)^1 + \ldots + {}^nc_n x^0 (1)^n$$

$$(1+x) = 1 + {}^nc_1 x^{n-1} + \ldots + {}^nc_n$$

Where n is considered for positive integer.

From the given value,

$$(27)^{999} = \frac{(28-1)^{999}}{7}$$

$$= \frac{28k-1}{7}$$

$$= \frac{28k-7+1-1}{7}$$

$$= \frac{7(4k-1)+6}{7}$$

Hence, the remainder is 6.

8. Given,

$$\frac{a+b}{2} = 5\sqrt{ab}$$

$$(a+b)^2 = 100ab \quad \ldots (1)$$

$$a^2 + b^2 + 2ab = 100ab$$

$$a^2 + b^2 - 2ab = 96ab$$

$$(a-b)^2 = 96ab \quad \ldots (2)$$

Divide equation (1) by (2).

$$\frac{(a+b)^2}{(a-b)^2}=\frac{100ab}{96ab}$$

$$\frac{(a+b)^2}{(a-b)^2}=\frac{25}{24}$$

$$\frac{(a+b)}{(a-b)}=\sqrt{\frac{25}{24}}$$

$$\frac{(a+b)}{(a-b)}=\frac{5\sqrt{6}}{12}$$

Hence, the correct option is (4).

9. The summation of n terms of series is,

$$S=\sqrt{3}+\sqrt{75}+\sqrt{243}+\sqrt{507}+\ldots\text{n terms}$$

$$=\sqrt{3}+\sqrt{3\times 25}+\sqrt{3\times 81}+\sqrt{3\times 169}+\ldots\text{n terms}$$

$$=\sqrt{3}\left(1+5+9+13+\ldots\text{n terms}\right)$$

Terms given in the bracket is AP with first term $a=1$ and the common difference is $d=4$.

The sum of n terms in AP is given by:

$$S=\sqrt{3}\times\frac{n}{2}\left(2+4\left(n-1\right)\right)=435\sqrt{3}$$

$$2n^2-n-435=0$$

$$n=\frac{1\pm 59}{4}$$

$$=15 \text{ or } -\frac{58}{4}$$

The valid value is only $n=15$.

10. Consider the function,

$$f\left(x\right)=\lim_{x\to 3}\frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$$

The given function is in indeterminate form that is $\frac{0}{0}$ form.

So,

Apply L'Hospital rule and differentiate numerator and denominator with respect to x.

$$f\left(x\right)=\lim_{x\to 3}\frac{\sqrt{3}\frac{1}{2}(x)^{\frac{1}{2}-1}-0}{\left(\frac{1}{2}(2x-4)^{\frac{1}{2}-1}\times 2\right)-0}$$

$$=\lim_{x\to 3}\frac{\frac{\sqrt{3}}{2\sqrt{x}}}{\frac{2}{2\sqrt{2x-4}}}$$

$$=\frac{1}{\sqrt{2}}$$

11. Consider the given curve equation,

$$x^2y^2 - 2x = 4(1-y)$$

Partially differentiate the above equation with respect to x.

$$\left(2xy^2 + 2yx^2 \cdot \frac{dy}{dx}\right) - 2(1) = 4(-1)\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2yx^2 + 4\right) = 2 - 2xy^2$$

$$\frac{dy}{dx} = \frac{2-2xy^2}{\left(2yx^2+4\right)}$$

The expression for the slope of the tangent is as follows,

$$m = \frac{dy}{dx}$$

$$= \frac{2-2xy^2}{\left(2yx^2+4\right)}$$

Tangent at point $(2,-2)$ is,

$$\left(\frac{dy}{dt}\right)_{(2,-2)} = \frac{2-2(2)(-2)^2}{\left(2(-2)(2)^2+4\right)}$$

$$= \frac{14}{12}$$

$$= \frac{7}{6}$$

Hence, the equation of tangent is given as:

$$(y+2)=m(x-2)$$

$$(y+2)=\frac{7}{6}(x-2)$$

$$7x-6y=26$$

Only (-2,-7) does not satisfy the above equation.

12. Consider the given equation,

$$y=\left[x+\sqrt{x^2-1}\right]^{15}+\left[x-\sqrt{x^2-1}\right]^{15} \quad \ldots\ldots (1)$$

First consider the term.

$$\left[x+\sqrt{x^2-1}\right]^{15}=t$$

Differentiate the above equation with respect to t.

$$\frac{dt}{dx}=15\left[x+\sqrt{x^2-1}\right]^{14}\cdot\left[1+\frac{x}{\sqrt{x^2-1}}\right]$$

$$\frac{dt}{dx}=\frac{15}{\sqrt{x^2-1}}\cdot\left[x+\sqrt{x^2-1}\right]^{15}$$

$$\frac{dt}{dx}=\frac{15}{\sqrt{x^2-1}}\cdot t$$

Rationalize the term $\left[x-\sqrt{x^2-1}\right]^{15}$,

$$\left[x-\sqrt{x^2-1}\right]^{15}=\frac{\left[x-\sqrt{x^2-1}\right]^{15}\times\left[x+\sqrt{x^2-1}\right]^{15}}{\left[x+\sqrt{x^2-1}\right]^{15}}$$

$$=\frac{\left(x^2-\left(x^2-1\right)\right)^{15}}{\left[x+\sqrt{x^2-1}\right]^{15}}$$

$$=\frac{1}{\left[x+\sqrt{x^2-1}\right]^{15}}$$

Simplify the above expression.

$$\left[x-\sqrt{x^2-1}\right]^{15}=\frac{1}{t}$$

The equation becomes:

$$y=\left(t+\frac{1}{t}\right)$$

$$\frac{dy}{dx}=\left(1-\frac{1}{t^2}\right)\times\frac{dt}{dx}$$

$$=\left(1-\frac{1}{t^2}\right)\times\frac{15}{\sqrt{x^2-1}}\cdot t$$

$$\sqrt{x^2-1}\frac{dy}{dx}=15\left(1-\frac{1}{t}\right)$$

Differentiate L.H.S and R.H.S.

$$\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{x}{\sqrt{x^2-1}}\frac{dy}{dx}=15\left(1+\frac{1}{t^2}\right)\times\frac{dt}{dx}$$

$$\frac{\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}}{\sqrt{x^2-1}}=15\left(1+\frac{1}{t^2}\right)\times\frac{15}{\sqrt{x^2-1}}\cdot t$$

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=225\left(t+\frac{1}{t}\right)$$

Rearrange the equation.

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=225y$$

13. The given equation is,

$$x^2+y^2-5x-y+5=0$$

Simplify the above equation.

$$\left(x-\frac{5}{2}\right)^2\left(\frac{-25}{4}\right)+\left(y-\frac{1}{2}\right)^2-\frac{1}{4}+5=0$$

$$\left(x-\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{3}{2}$$

Consider a point Q on circle such that. $\mathrm{Q}=\left(\frac{5}{2}+\sqrt{\frac{3}{2}}\cos q,\frac{1}{2}+\sqrt{\frac{3}{2}}\sin q\right)$.

Thus,

$$(\mathrm{PQ})^2=\left(\frac{5}{2}+\sqrt{\frac{3}{2}}\cos q\right)^2+\left(\frac{1}{2}+\sqrt{\frac{3}{2}}\sin q\right)^2$$

$$=14+5\sqrt{\frac{3}{2}}(\cos q+\sin q)$$

Maximum $(\mathrm{PQ})^2$ is,

$$(\mathrm{PQ})^2=14+5\sqrt{\frac{3}{2}}\left(\sqrt{2}\right)$$

$$=14+5\sqrt{3}$$

14. Consider the given integral,

$$I=\int\sqrt{1+2\cot x(\operatorname{cosec} x+\cot x)}dx$$

Simplify the above integral.

$$I=\int\sqrt{1+2\cot x\cos ecx+2\cot^2 x}\ dx$$

$$=\int\sqrt{1+2\frac{\cos x}{\sin^2 x}+2\frac{\cos^2 x}{\sin^2 x}}\ dx$$

$$=\int\sqrt{\frac{\sin^2 x+2\cos x+2\cos^2 x}{\sin^2 x}}\ dx$$

Further simplify the above equation.

$$I = \int \sqrt{\frac{(1+\cos x)^2}{\sin^2 x}}dx$$

$$= \int \frac{1+\cos x}{\sin x}d$$

$$= \int \frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}dx$$

$$= \int \cot\left(\frac{x}{2}\right)$$

Integrate the above equation.

$$I = 2\log\left(\sin\frac{x}{2}\right) + C$$

15. Consider the given integral,

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{(\tan x + \cot x)^3}\, dx$$

Simplify the above integral.

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{(\tan x+\cot x)^3}dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)^3}dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{\left(\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right)^3}dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{\left(\frac{1}{\sin(2x/2)}\right)^3}dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \cos 2x\sin 2x\sin^2 2xdx$$

Further, simplify the above equation.

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{(\tan x+\cot x)^3}dx = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (2\cos 2x\sin 2x)\sin^2 2xdx$$

$$= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (\sin 4x)\sin^2 2xdx$$

$$= \frac{1}{4}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (\sin 4x)(1-\cos^2 4x)dx$$

Simplify the above equation.

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{(\tan x+\cot x)^3}dx=\frac{1}{4}\left(\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\sin 4x dx-\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\sin 8x dx\right)$$

$$=\frac{1}{4}\left(-\frac{\cos 4x}{4}\right)_{\frac{\pi}{12}}^{\frac{\pi}{4}}-\left(-\frac{\cos 8x}{8}\right)_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

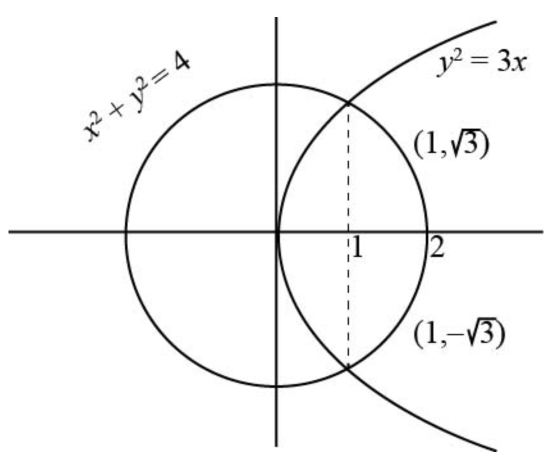
$$=\frac{1}{4}\left[-\frac{\cos 4x}{4}+\frac{\cos 8x}{8}\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$=\frac{1}{4}\left[-\frac{\cos\left(4\frac{\pi}{4}\right)}{4}+\frac{\cos\left(8\frac{\pi}{4}\right)}{8}+\frac{\cos\left(4\frac{\pi}{4}\right)}{4}-\frac{\cos\left(8\frac{\pi}{4}\right)}{8}\right]$$

Simplify the above equation further.

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8\cos 2x}{(\tan x+\cot x)^3}dx=\frac{15}{128}$$

16. Consider the diagram:



The curve $x^2 + y^2 = 4$ and $y^2 = 3x$ is given below.

$$x^2 + y^2 = 4$$

$$x^2 + 3x - 4 = 0$$

$$x = -4 \text{ or } x = 1$$

For $x = -4, y^2$ is negative. Hence, $x = 1$ is the intersection point of two curves.

Then at $x = 1, y = \pm\sqrt{3}$, the intersection points are $\left(1, \sqrt{3}\right)$ and $\left(1, -\sqrt{3}\right)$.

Area of the smaller portion is given as:

$$A = \int_0^1 \sqrt{3} \cdot \sqrt{x} dx + \int_0^1 \left(\int_0^1 \sqrt{3} \cdot \sqrt{x} dx + \int_1^2 \sqrt{4 - x^2} dx \right) \cdot 2$$

$$= 2 \times \left(\sqrt{3} \left(\frac{x^{3/2}}{3/2} \right)_0^1 + \left(\frac{x}{2} \sqrt{4 - x^2} + 2 \sin^{-1} \left(\frac{x}{2} \right) \right)_1^2 \right)$$

$$= 2 \times \left(\sqrt{3} \left(\frac{2}{3} \right) + \left(2 \times \frac{\pi}{2} - \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) \right) \right)$$

$$= \frac{1}{\sqrt{3}} + \frac{4\pi}{3}$$

17. Simplify the given differential equation.

$$ydx-\left(x+3y^{2}\right)dy=0$$

$$\frac{dy}{dx}=\frac{x}{y}+3y$$

$$\frac{dy}{dx}-\frac{x}{y}=3y$$

Integrate the above factor.

$$IF=e^{-\int\frac{1}{y}dy}$$

$$=e^{-\ln y}$$

$$=\frac{1}{y}$$

Hence, the solution is,

$$\frac{x}{y}=\int 3y\cdot IFdy$$

$$=\int 3y\times\frac{1}{y}dy$$

$$=3y+\mathrm{C}$$

Satisfying the given point in equation:

$$1=3+\mathrm{C}$$

$$\mathrm{C}=-2$$

Hence, the solution is,

$$x=3y^{2}-2y$$

Thus, the option (2) satisfies the solution.

18. The given equation of straight lines is,

$$tx-2y-3t=0 \quad \dots (1)$$

$$x-2ty+3=0 \quad \dots (2)$$

Multiply equation (1) and (2) with t:

$$t^2x-2ty-3t^2=0 \quad \dots. (3)$$

$$tx-2t^2y+3t=0 \quad \dots (4)$$

Subtract equation (1) with equation(4).

$$y=\frac{3t}{t^2-1} \Rightarrow 2y=-3\left(\frac{2t}{1-t^2}\right)=-3\tan 2A$$

Subtract equation (2) with (4) gives:

$$x=\frac{3t^2+1}{t^2-1}=-3\sec 2A$$

$$\sec^2(2A)-\tan^2(2A)=1$$

$$\frac{x^2}{9}-\frac{y^2}{9/4}=1$$

Simplify the above equation:

$$\lambda(T\cdot A)=2b$$

$$=2\times\frac{3}{2}$$

$$=3$$

The eccentricity is calculated as,

$$e^2 = 1 + \frac{\left(\frac{9}{4}\right)}{9}$$

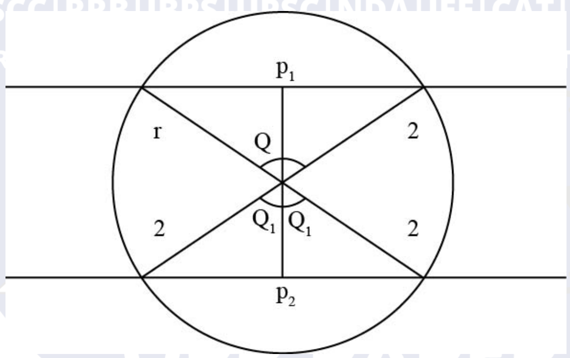
$$= 1 + \frac{1}{4}$$

$$= \frac{5}{4}$$

$$e = \frac{\sqrt{5}}{2}$$

Hence, the correct option is 4.

19. The diagram is shown below according to question.



From figure,

$$\cos 2A = \frac{1}{7}$$

$$\cos^2 A - 1 = \frac{1}{7}$$

$$\cos^2 A = \frac{4}{7}$$

Simplify the above equation.

$$\frac{d_1^2}{4} = \frac{4}{7}$$

$$d_1 = \frac{4}{\sqrt{7}}$$

For $\sec 2B$,

$$\sec 2B = 7$$

$$\frac{1}{2\cos^2 A - 1} = 7$$

$$2\cos^2 B - 1 = \frac{1}{7}$$

Simplify the above equation.

$$\cos^2 B = \frac{4}{7}$$

$$\frac{d_2^2}{4} = \frac{4}{7}$$

$$d_2 = \frac{4}{\sqrt{7}}$$

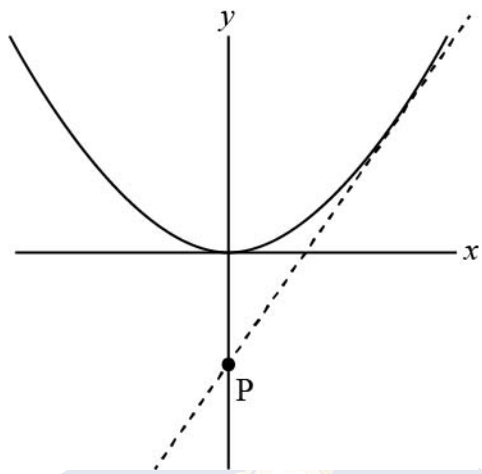
The total distance between these chords is d. Hence,

$$d = d_1 + d_2$$

$$= \frac{4}{\sqrt{7}} + \frac{4}{\sqrt{7}}$$

$$= \frac{8}{\sqrt{7}}$$

20. Consider the required diagram:



The standard equation of the parabola is,

$$x^2 = 4ay$$

Condition of the tangency is given as:

$$c^2 = a^2\left(1+m^2\right)$$

If $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, then tangent to $x^2 + y^2 = 4$ is,

$$y = mx \pm 2\sqrt{1+m^2}$$

Also $x^2 = 4y$

$$x^2 = 4mx + 8\sqrt{1+m^2}$$

$$x^2 - 4mx - 8\sqrt{1+m^2} = 0$$

For D = 0, the above equation becomes,

$$D = b^2 - 4ac$$

$$= 0$$

Substitute the value of a, b and c in the above expression

$$16m^2+4\times 8\sqrt{1+m^2}=0$$

$$m^2=2\sqrt{1+m^2}$$

$$m^4-4m^2-4=0$$

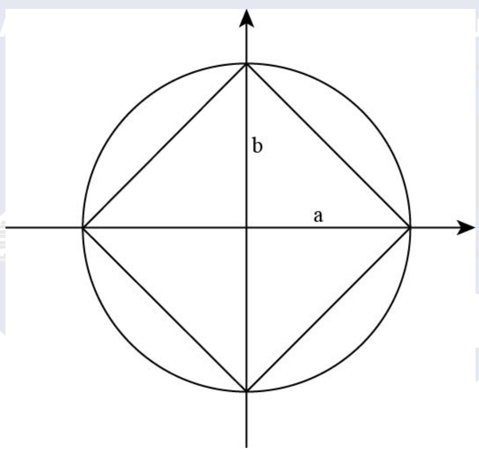
$$\left(m^2\right)^2-4\left(m^2\right)-4=0$$

The root of the above equation is,

$$m^2=2+2\sqrt{2}$$

$$=2\left(\sqrt{2}+1\right)$$

21. Consider the required diagram:



Given eccentricity of an ellipse $\frac{3}{5}$.

$$e=\frac{3}{5}$$

$$2ae=6$$

$$a=5$$

The equation of an ellipse is,

$$b^2 = a^2\left(1 - e^2\right)$$

$$b^2 = 16$$

$$b = 4$$

Area of an ellipse is given as:

$$A = 4 \times \frac{1}{2} ab$$

$$= 40$$

22. Normal to given lines equation is,

$$n = n_1 \times n_2$$

$$= \begin{vmatrix} i & j & k \\ 6 & 7 & 8 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= 9i - 18j + 9k$$

Simplify the above equation:

$$n = i - 2j + k$$

The plane containing given lines are,

$$1(x+1) - 2(y-1) + 1(z-3) = 0$$

$$x - 2y + z = 0$$

Normal line form $(1, -2, 1)$,

$$\frac{x_1 - 1}{1/\sqrt{6}} = \frac{y_1 + 2}{-2/\sqrt{6}} = \frac{z_1 - 1}{1/\sqrt{6}} = k \quad \dots (1)$$

This point lies on the plane. Substitute (x_1, y_1, z_1) in terms of k in the plane equation.

$$k = -\sqrt{6}$$

Substitute the value of k in equation (1) to find the value of x_1, x_2 and x_3.

$$\underbrace{\frac{x_1 - 1}{1/\sqrt{6}}}_{\text{I}} = \underbrace{\frac{y_1 + 2}{-2/\sqrt{6}}}_{\text{II}} = \underbrace{\frac{z_1 - 1}{1/\sqrt{6}}}_{\text{III}} = \underbrace{-\sqrt{6}}_{\text{IV}}$$

Consider I and IV,

$$x_1 = 0$$

Consider II and IV,

$$y_1 = 0$$

Consider III and IV,

$$z_1 = 0$$

Hence, the coordinates is,

$$(x_1, y_1, z_1) = (0,0,0)$$

23. The equation of line of intersection is given by:

$$n = n_1 \times n_2$$

$$= \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= 2i - 7j - 13k$$

Now the equation is given as:

$$3x - y + z = 1$$
$$x + 4y - 2z = 2$$

But $z = 0$ hence,

$$x = \frac{6}{13}, y = \frac{5}{13}$$

So the line of equation is given as:

$$\frac{x - \frac{6}{13}}{2} = \frac{y - \frac{5}{13}}{-7} = \frac{z}{-13}$$

24. The product of diagonal is given as:

$$\mathrm{d}_1 \times \mathrm{d}_2 = \begin{vmatrix} i & j & k \\ 8 & -6 & 0 \\ 3 & 4 & -12 \end{vmatrix}$$
$$= 72i + 96j + 50k$$

Area of the parallelogram is given as,

$$A = \frac{1}{2}\left|\mathrm{d}_1 \times \mathrm{d}_2\right|$$
$$= \sqrt{72^2 + 96^2 + 50^2}$$
$$= 65 \text{ sq.units}$$

25. The average of age of 25 teachers is given as,

$$\frac{x_1 + x_2 + x_3 + x_4 + \dots\dots\dots\dots + x_{25}}{25} = 40$$

$$x_1 + x_2 + x_3 + x_4 + \dots\dots\dots\dots + x_{25} = 1000 \qquad \dots (1)$$

Consider the age of new teacher is y years.

$$\frac{x_1 + x_2 + x_3 + x_4 + \dots\dots\dots\dots + x_{25} - 60 + y}{25} = 39$$

From the equation (1),

$$1000 - 60 + y = 39 \times 25$$

$$y = 35$$

26. The probability that the target is hit by P, Q and R is,

$$P = (\text{P hits}) \cup (\text{Q hits}) \cup (\text{P and Q hit})$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{8}\right)$$

$$= \frac{21}{64}$$

27. Probability to obtain at least one head and at least one tail is calculated as below,

$$P = 1 - \left(P(\text{all heads}) + P(\text{all tails})\right)$$

$$= 1 - \left(\frac{1}{2^8} + \frac{1}{2^8}\right)$$

$$= 1 - \frac{1}{2^7}$$

$$= \frac{127}{128}$$

28. According to given question:

$$S = \left\{ x \in [0, 2\pi] : \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix} = 0 \right\}$$

$$\cos^3 x - \sin^3 x = 0$$

$$\tan^3 x = 1$$

$$\tan x = 1$$

The relation of x is,

$$x \in \left\{ \frac{\pi}{4}, \frac{5\pi}{4} \right\}$$

Then,

$$\sum_{x\in S}\tan\left(\frac{\pi}{3}+x\right)=\sum_{x\in S}\frac{\tan\frac{\pi}{3}+\tan x}{1-\tan\frac{\pi}{3}\tan x}$$

$$=\sum_{x\in S}\frac{1+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{3}}$$

There are only two value of set. Hence, the sum is given as:

$$S=2\times\frac{\sqrt{3}+1}{1-\sqrt{3}\times 1}$$
$$=2\times\frac{\sqrt{3}+1}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$=-4-2\sqrt{3}$$

29. Consider $x^2=\cos 2A$,

$$A=\frac{1}{2}\cos^{-1}x^2$$

According to question;

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]=\tan^{-1}\left[\frac{\sqrt{1+\cos 2A}+\sqrt{1-\cos 2A}}{\sqrt{1+\cos 2A}-\sqrt{1-\cos 2A}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{1+\left(2\cos^2 A-1\right)}+\sqrt{1-\left(1-2\sin^2 A\right)}}{\sqrt{1+\left(2\cos^2 A-1\right)}-\sqrt{1-\left(1-2\sin^2 A\right)}}\right]$$

$$=\tan^{-1}\left[\frac{1+\tan A}{1-\tan A}\right]$$

$$=\tan^{-1}\left(\frac{\pi}{4}+A\right)$$

Further simplify the above equation.

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

30. The proposition of $(\sim p)\vee(p\wedge\sim q)$ is equivalent to:

p	q	$\sim p$	$\sim q$	$p\wedge\sim q$	$(\sim p)\vee(p\wedge\sim q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	F	F

Which is same as $p\rightarrow\sim q$.